1. Each of the following statements is either true or false. If the statement holds in general, write "`True`". Otherwise, write "`False`". **No justification necessary.**

   (a) Given a set X, there exists a set of strictly larger cardinality.
   (b) The set of all finite subsets of $\mathbb{Q}$ is countable.
   (c) The set of all finite topological spaces (up to homeomorphism) is countable.
   (d) The set $\{U \subseteq \mathbb{R} \mid U \text{ is infinite}\} \cup \{\varnothing\}$ is a topology on $\mathbb{R}$.
   (e) Consider $[0,2]$ as a subspace of $\mathbb{R}$ with the standard topology. The subset $(1,2] \subseteq [0,2]$ is open.
   (f) Consider $[0,2]$ as a subspace of $\mathbb{R}$ with the standard topology. Then $(0,1) \cup \{2\} \subseteq [0,2]$ is open.
   (g) Let $f : X \to \mathbb{R}$ be a map from a space X to $\mathbb{R}$ (with the standard topology). Then X is continuous if and only if $f^{-1}(B_r(x))$ is open for every **rational** numbers $x, r \in \mathbb{Q}$, $r > 0$.
   (h) Let X be a space with the property that points $\{x\}$ are **open**. Then X is a T_1-space.
   (i) The Cartesian product of two quotient maps is a quotient map.
   (j) The composite of two quotient maps is a quotient map.
   (k) The set $\{a, b, c, d\}$ with the topology $\{\varnothing, \{c\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}\}$ is connected.

2. Each of the following statements is either true or false. If the statement holds in general, write "`True`". Otherwise, state a counterexample. **No justification necessary.**

   Note: You can get partial credit for correctly writing "`False`" without a counterexample.

   (i) Let $f : X \to Y$ be a function of sets X and Y. If there is a function g so that $g \circ f : X \to X$ is the identity map of X, then f is invertible.
   (ii) Let X be a topological space, and $S \subseteq X$ be a subset with no limit points. Then S is closed.
   (iii) Let X be a topological space, and $C \subseteq X$ a closed set. The inclusion map $C \to X$ is a closed map.
   (iv) Let A be a subset of a metric space (X, d). Then any element of ∂A must be both a limit point of A, and a limit point of $X \setminus A$.
   (v) Let X be a topological space, and $S \subseteq X$. Then $\partial S = \partial\left(\overline{S}\right)$.
   (vi) Let X be a topological space, and $S \subseteq X$. Then $\overline{S} = X \setminus \text{Int}(X \setminus S)$.
   (vii) If $A \subseteq B$, then $\text{Int}(A) \subseteq \text{Int}(B)$
   (viii) If $A \subseteq B$, then all limits points of A are also limit points of B.
   (ix) If $\text{Int}(A) = \text{Int}(B)$ and $\overline{A} = \overline{B}$, then $A = B$.
   (x) If $\text{Int}(A) = \overline{A}$, then A is both open and closed.
   (xi) Let $(X, \mathcal{T})$ be a topological space, and let x, y be **distinct** points in X. Let $(a_n)_{n\in\mathbb{N}}$ be the sequence $x\ y\ x\ y\ x\ y\ x\ y \cdots$. Then $(a_n)_{n\in\mathbb{N}}$ does not converge.
   (xii) If a sequence of points $(a_n)_{n\in\mathbb{N}}$ in a topological space X converges to a point a_∞, then a_∞ is a limit point of the set $\{a_n \mid n \in \mathbb{N}\}$.
   (xiii) Suppose $(a_n)_{n\in\mathbb{N}}$ is a sequence in a topological space, and that x is a limit point of the set $\{a_n \mid n \in \mathbb{N}\}$. Then there is some subsequence converging to x.
   (xiv) There is no sequence in $\mathbb{R}$ (with the standard topology) with the property that, for any $r \in \mathbb{R}$, there is some subsequence converging to $\mathbb{R}$.
   (xv) Let X and Y be metric spaces, and $f : X \to Y$ a continuous function. If $B \subseteq X$ is bounded, then $f(B)$ is bounded.
   (xvi) Let $f : X \to Y$ be continuous function. If X is Hausdorff, then $f(X)$ is Hausdorff.

(xvii) Let $f : X \to Y$ be a continuous map. Then the restriction of f to any subspace of X is continuous (with respect to the subspace topology).

(xviii) Let $(f_n : X \to \mathbb{R})_{n \in \mathbb{N}}$ be a sequence of continuous functions that converge pointwise to a function $f : X \to \mathbb{R}$. If f is continuous, then the functions must converge uniformly.

(xix) An injective quotient map is necessarily a homeomorphism.

(xx) Let $p : X \to A$ be a quotient map. If X is a T_1-space, then A is a T_1-space.

(xxi) If $(X, \mathcal{T})$ is a connected topological space, and $\mathcal{T}'$ is a coarser topology on X , then $(X, \mathcal{T}')$ is also connected.

(xxii) Let A and B be nonempty subsets of a topological space $(X, \mathcal{T})$. If A and B are connected and $A \cap B$ is nonempty, then $A \cap B$ is connected.

3. Let X be a set, and suppose that $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on X.
   (a) Show that the intersection $\mathcal{T}_1 \cap \mathcal{T}_2$ is a topology on X.
   (b) Show by example that the union $\mathcal{T}_1 \cup \mathcal{T}_2$ need not be a topology on X.

4. Let $(X, \mathcal{T})$ be a topological space, and let $S \subseteq X$ be a subset. Show that $\partial(\partial S) = \partial S$ if and only if ∂S has empty interior.

5. Let $(X, \mathcal{T})$ be a topological space, and let $S \subseteq X$ be a subset. Suppose that the subspace topology on S is the discrete topology. Prove or give a counterexample: S is closed as a subset of X.

6. Consider the following functions $f : \mathbb{R} \to \mathbb{R}$.
   - $f(x) = x$
   - $f(x) = x^2$
   - $f(x) = x + 1$
   - $f(x) = 0$
   - $f(x) = \cos(x)$
   - $f(x) = -x$

   Determine whether these functions are continuous ...
   (a) ... when $\mathbb{R}$ has the topology $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\varnothing\} \cup \{\mathbb{R}\}$.
   (b) ... when $\mathbb{R}$ has the cofinite topology.
   (c) ... when $\mathbb{R}$ has the cocountable topology.
   (d) ... when $\mathbb{R}$ has the topology $\mathcal{T} = \{\mathbb{R}\} \cup \{U \subseteq \mathbb{R} \mid 0 \notin U\}$.
   (e) ... when $\mathbb{R}$ has the topology $\mathcal{T} = \{\varnothing\} \cup \{U \subseteq \mathbb{R} \mid 0 \in U\}$.

7. Consider the set $X = \{a, b, c, d\}$ with the topology

$$\mathcal{T} = \{\varnothing, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}.$$

   (a) Write down a permutation of the elements of X that is not continuous.
   (b) Is there a nonconstant, nonidenty map $X \to X$ that is continuous?
   (c) Is there a non-identity permutation of the elements of X that is a homeomorphism?
   (d) Let $[0, 1]$ be the closed interval with the standard topology. Is there a non-constant continuous map $[0, 1] \to X$?
   (e) Is there a non-constant continuous map $X \to [0, 1]$?

8. Prove the following propositions.
   (a) A function $f : X \to Y$ of topological space is *continuous at the point* $x \in X$ if for every neighbourhood U of $f(x)$, there is a neighbourhood V of x such that $f(V) \subseteq U$.

   Proposition. A function $f : X \to Y$ of topological spaces is continuous if and only if it is continuous at every point $x \in X$.

(b) **Proposition.** Let $f : X \to Y$ be a function of topological spaces. Then f is continuous if and only if $f^{-1}(\text{Int}(B)) \subseteq \text{Int}(f^{-1}(B))$ for every $B \subseteq Y$.

(c) **Proposition.** Let $f : X \to Y$ be a function of topological spaces, and let $\mathcal{S}$ be a subbasis for Y. Then f is continuous if and only if $f^{-1}(U)$ is open for every subbasis element $U \subseteq \mathcal{S}$.

9. Let $\{0, 1\}$ be a topological space with the discrete topology. Let X be a topological space and $A \subseteq X$ a subset. Define the *characteristic function on A*

$$\chi_A : X \longrightarrow \{0,1\}$$
$$\chi_A(x) = \begin{cases} 1, x \in A \\ 0, x \notin A \end{cases}$$

   (a) Prove that χ_A is continuous at a point $x \in X$ if and only if $x \notin \partial A$.

   (b) Prove that χ_A is continuous if and only if A is both open and closed.

10. You proved on Homework #6 that if $X = A \cup B$ for closed sets A and B, and $f : X \to Y$ is a function whose restriction to A and B are both continuous, then f is continuous. Prove or disprove that the same statement holds if X is a union of (possibly infinitely many) closed sets A_i, with $f|_{A_i}$ continuous for each i.

11. Let (X, d) be a metric space, and let $\mathcal{B}$ be a basis for the topology $\mathcal{T}_d$ induced by d.

   (a) Let $S \subseteq X$ be a subset, and $s \in S$. Show that s is an interior point of S if and only if there is some element $B \in \mathcal{B}$ such that $s \in B$ and $B \subseteq S$.

   (b) Deduce that $\text{Int}(S) = \bigcup_{B \in \mathcal{B}, B \subseteq S} B$.

12. Let $f : X \to Y$ be a function of topological spaces. Prove that f is open if and only if $f(\text{Int}(A)) \subseteq \text{Int}(f(A))$ for all sets $A \subseteq X$.

13. (a) Let $(X, \mathcal{T}_X)$ be a Hausdorff topological space, and let $x_1, \ldots, x_n$ be a finite collection of points in X. Show that there are open sets $U_1, \ldots, U_n$ such that $x_i \in U_i$, and which are pairwise disjoint (this means $U_i \cap U_j = \varnothing$ for all $i \neq j$).

   (b) Let X be a finite topological space. Prove if X is Hausdorff, then it has the discrete topology.

14. Let $A \subseteq \mathbb{R}$ be a nonempty bounded subset. Prove that its supremum $\sup(A)$ is either in the set A, or it is a limit point of A.

15. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of points in a topological space X that converges to $a_\infty \in X$.

   (a) Prove that, if we replace the topology on X with any coarser topology, then the sequence $(a_n)_{n \in \mathbb{N}}$ will still converge to a_∞.

   (b) Show by example that, if we replace the topology on X with any finer topology, then the sequence $(a_n)_{n \in \mathbb{N}}$ may no longer converge to a_∞.

16. Let X be a metric space, and let $a_\infty \in X$. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence in X with the property that every subsequence of $(a_n)_{n \in \mathbb{N}}$ has a subsequence that converges to a_∞. Prove that $(a_n)_{n \in \mathbb{N}}$ converges to a_∞.

17. Let X be a topological space $A \subseteq X$, and $h : X \to \mathbb{R}$ a continuous function. Suppose there is a constant c such that $h(x) \leq c$ for all $x \in A$. Prove that $h(x) \leq c$ for all $x \in \overline{A}$.

18. Suppose that X and Y are nonempty topological spaces, and that the product topology on $X \times Y$ is Hausdorff. Prove that X and Y are both Hausdorff.

19. (a) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $f : X \to Y$ be a function. Recall that the *graph* of f is defined to be the subset of $X \times Y$

$$\{\ (x, f(x)) \in X \times Y \mid x \in X\ \}.$$

Suppose that Y is Hausdorff. Show that, if f is continuous, then the graph of f is a closed subset of $X \times Y$ with respect to the subspace topology $\mathcal{T}_{X\times Y}$.

(b) Show by example that, if Y is not Hausdorff, the graph of a function $f : X \to Y$ need not be closed.

20. Let $f, g : X \to Y$ be continuous maps between topological spaces. Show that if Y is Hausdorff, then the set $S = \{x \in X \mid f(x) = g(x)\}$ is closed.

21. (a) Let $S \subseteq X$ be a subset of a topological space X. Explain why, to show that S is closed, it suffices to show that S is the preimage of a closed set under a continuous function.

(b) Show that the following subsets of $\mathbb{R}^2$ (with the usual topology) are closed:

$$\{(x, y) \mid xy = 1\} \qquad S^1 = \{(x, y) \mid x^2 + y^2 = 1\} \qquad D^2 = \{(x, y) \mid x^2 + y^2 \leq 1\}$$

22. Let X be a topological space.

(a) Suppose that X is Hausdorff. Let $x \in X$. Show that the intersection of all open sets containing x is equal to $\{x\}$.

(b) Show that the converse statement does not hold. Specifically, suppose $(X, \mathcal{T})$ is a infinite set with the cofinite topology. Show that $(X, \mathcal{T})$ is not Hausdorff, but for any $x \in X$ the intersection of all open sets containing X is equal to $\{x\}$.

23. Let $\mathbb{R}^\omega = \prod_{\mathbb{N}} \mathbb{R}$ (so an element of $\mathbb{R}^\omega$ is precisely a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers.) Fix sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ with $a_n \neq 0$ for all n. Define

$$\begin{aligned} h : \mathbb{R}^\omega &\longrightarrow \mathbb{R}^\omega \\ (x_1, x_2, x_3, \ldots) &\longmapsto (a_1x_1 + b_1, a_2x_2 + b_2, a_3x_3 + b_3, \ldots) \end{aligned}$$

Determine whether h is a homeomorphism if its domain and codomain are given the product topology, and if they are given the box topology.

24. Find an explicit homeomorphism bewteen the intervals $(0, 1)$ and $(1, \infty)$ with the Euclidean metric.

25. Let (X, d) be a metric space. Show that the function $D(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}$ defines a new metric on X.

26. Let $f_n : X \to Y$ be a sequence of functions from a topological space X to a metric space (Y, d). Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in X converging to x_∞.

(a) Show that, if the sequence $\big(f_n\big)_{n\in\mathbb{N}}$ converges uniformly to a function f_∞, then the sequence $\big(f_n(x_n)\big)_{n\in\mathbb{N}}$ converges to $f_\infty(x_\infty)$.

(b) Show that this conclusion need not hold if the sequence of functions $\big(f_n\big)_{n\in\mathbb{N}}$ only converges pointwise to f_∞.

27. Consider the sequence of functions $f_n : [0.1] \to [0, 1]$ defined by the equations $f_n(x) = \begin{cases} nx, & 0 \leq x \leq \frac{1}{n} \\ 1, & \frac{1}{n} \leq x \leq 1. \end{cases}$

(a) Show that this sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise to the constant function $f(x) = 1$.

(b) Show that this sequence does not converge uniformly.

(c) Conclude that, even when a sequence of continuous functions converges pointwise to a continuous function, the convergence need not be uniform.

28. Let (X, d) be a metric space. Fix $x_0 \in X$ and $r > 0$ in $\mathbb{R}$.
    (a) Show that the closure of the open ball $B_r(x_0)$ is contained in $\{x \mid d(x_0, x) \leq r\}$.
    (b) Give an example of a metric space where the closure is always equal to this set.
    (c) Give an example of a metric space X and a ball $B_r(x_0)$ whose closure is a strict subset of $\{x \mid d(x_0, x) \leq r\}$.
    (d) Show that the set $\{x \mid d(x_0, x) \leq r\}$ is closed in any metric space.

29. Let (X, d) be a metric space, and let $S \subseteq X$ be a **finite** subset of X. Prove that S ...
    (a) is closed.
    (b) is bounded.
    (c) has no limit points.
    (d) Show that S may or may not have empty interior $\mathring{S} = \varnothing$.

30. Given an index set I, define metrics on $\mathbb{R}^I$ as follows. For points $\mathbf{x} = (x_i)_{i \in I}$ and $\mathbf{y} = (y_i)_{i \in I}$, recall that we defined the uniform metric by

$$d_U(\mathbf{x}, \mathbf{y}) = \sup_{i \in I} \Big\{ \min\{|x_i - y_i|, 1\} \Big\}.$$

Moreover, when $I = \mathbb{N}$, define a new metric let

$$d(\mathbf{x}, \mathbf{y}) = \sup_{i \in \mathbb{N}} \left\{ \frac{\min\{|x_i - y_i|, 1\}}{i} \right\}.$$

    (a) Verify that the uniform metric d_U is in fact a metric. We proved that this metric induces the uniform topology on $\mathbb{R}^I$.
    (b) Suppose $I = \mathbb{N}$. Verify that d is a metric.
    (c) Show that d induces the product topology on $\mathbb{R}^\omega$.

31. Let (X, d) be a metric space. Show that the map $d : X \times X \to \mathbb{R}$ is continuous with respect to the product topology on X and the standard topology on $\mathbb{R}$. Show moreover that the topology on X induced by d is the coarsest topology making d continuous.

32. Show that metrizability is a topological property. In other words, show that, if X and Y are homeomorphic topological spaces, then X is metrizable if and only if Y is.

33. Let X be a finite set (of, say, n elements), and let d be a metric on X. What is the topology $\mathcal{T}_d$ on X induced by d? Show in particular that this topology will be the same for every possible metric d.

34. We proved in class that, if d and $\tilde{d}$ are two metrics on a set X, then d and $\tilde{d}$ are topologically equivalent if they satisfy the following condition: For each $x \in X$, there exist positive constants $\alpha, \beta > 0$ such that for every $y \in X$,

$$\alpha\, d(x, y) \leq \tilde{d}(x, y) \leq \beta\, d(x, y).$$

(Note that α and β depend on x but are independent of y.) Show that the converse of this statement fails: find a set X and equivalent metrics d and $\tilde{d}$ on X that fail this condition.

35. Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose that $f : X \to Y$ is a function that preserves distances in the sense that

$$d_Y\Big(f(x_1), f(x_2)\Big) = d_X(x_1, x_2) \qquad \text{for all } x_1, x_2 \in X.$$

Show that f is continuous, and is an embedding of topological spaces. Such maps are called *isometric embeddings*.

36. Let J be an uncountable index set, and consider the product $\mathbb{R}^J$ with the product topology associated to the standard topology on $\mathbb{R}$. In this question, we will show that this product is Hausdorff but not metrizable. Define tthe subset $A = \{(x_j)_{j \in J} \mid x_j = 1 \text{ for all by finitely many } j \in J\ \}$ of $\mathbb{R}^J$. Let $\mathbf{0}$ denote the element $(0)_{j \in J} \in \mathbb{R}^J$ that is constant 0 in every component.

(a) Consider an arbitrary basis element $U = \prod_{j \in J} U_j$ for the product topology on $\mathbb{R}$ with $\mathbf{0} \in U$. Explain why U contains an element of A. Conclude that $\mathbf{0} \in \overline{A}$.

(b) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points $a_n = (a_{n,j})_{j\in J} \in A$. Explain why there is an index $j \in J$ such taht $a_{n,j} = 1$ for every n. *Hint:* J is uncountable.

(c) Construct a neighourhood $V = \prod_{j\in J} V_j$ of $\mathbf{0}$ that does not contain a_n for any $n \in \mathbb{N}$. Deduce that the sequence $(a_n)_{n\in\mathbb{N}}$ does not converge to $\mathbf{0}$.

(d) Conclude that the product $\mathbb{R}^J$ is Hausdorff, but not metrizable.

37. Let X be a topological space, and X^* a quotient space of X. Show that X^* is a T_1–space if and only if every equivalence class in X^* is closed as a subset of X.

38. Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the projection onto the first factor. Show that the restriction of π_1 to the subset

$$\{(x, y) \mid xy = 1\} \cup \{(0, 0)\}$$

is continuous and surjective, but is not a quotient map.

39. Let $p : X \to Y$ be a quotient map of topological spaces, and $A \subseteq X$ a subset. Show that, if p is an open map, then the restriction of p to A is also a quotient map.

40. Let $X \subseteq \mathbb{R}^2$ be the subspace $\{(x, n) \mid n \in \mathbb{N}, x \in [0, 1]\}$ consisting of the horizontal line $[0, 1] \times \{n\}$ for each natural number n. Let $Y \subseteq \mathbb{R}^2$ be the subspace $\{(x, \frac{x}{n}) \mid n \in \mathbb{N}, x \in [0, 1]\}$ consisting of the line of slope $\frac{1}{n}$ through the origin for each natural number n. Define a map $g : X \to Y$ by $g(x, n) = (x, \frac{x}{n})$.

(a) Verify that g is continuous and surjective.

(b) Determine whether g is a quotient map.

41. Let X be a topological space and $Y \subseteq X$. Show that a separation of Y is precisely a pair of disjoint nonempty set $A, B \subseteq Y$ whose union is Y, such that neither set contains a limit point of the other.

42. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of connected subsets of a space X. Suppose that $A_n \cap A_{n+1} \neq \varnothing$ for each n. Show that the union $\bigcup_n A_n$ is connected.

43. Let $(X, \mathcal{T})$ be a topological space. Let $\{A_n \mid n \in \mathbb{N}\}$ be a family of connected subspaces in X such that $A_{n+1} \subseteq A_n$ for every $n \in \mathbb{N}$. Is $\bigcap_{n\in\mathbb{N}} A_n$ is necessarily connected?

44. Let $(X, \mathcal{T})$ be a topological space, and let $A, B \subseteq X$. Suppose $A \cup B$ and $A \cap B$ are connected. Prove that if A and B are both closed or both open, then A and B are connected.

45. **Definition (Adherent sets).** Let $(X, \mathcal{T}_X)$ be a topological space, and let $A, B \subseteq X$. Then A and B are called *adherent* if

$$(A \cap \overline{B}) \cup (\overline{A} \cap B) \neq \varnothing.$$

(a) Give examples of disjoint adherent subsets of $\mathbb{R}$ (with the Euclidean metric).

(b) Let $(X, \mathcal{T}_X)$ be a topological space and $A, B, C \subseteq X$. Prove or give a counterexample: if A and B are adherent, and B and C are adherent, then A and C are adherent.

(c) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $f : X \to Y$ be a continuous map. Prove that, if A and B are adherent subsets of X, then $f(A)$ and $f(B)$ are adherent subsets of Y.

(d) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be Hausdorff topological spaces. Suppose that $f : X \to Y$ has the property that, whenever A and B are adherent subsets of X, then $f(A)$ and $f(B)$ are adherent subsets of Y. Prove that f is continuous.