1. Each of the following statements is either true or false. If the statement holds in general, write "`True`". Otherwise, write "`False`". **No justification necessary.**

   (a) Given a set X, there exists a set of strictly larger cardinality.

   True. (Hint: Consider the power set of X.)

   (b) The set of all finite subsets of $\mathbb{Q}$ is countable.

   True. (Hint: Write it as a countable union of countable sets.)

   (c) The set of all finite topological spaces (up to homeomorphism) is countable.

   True. (Hint: The possible topologies on X are a subset of the power set of X.)

   (d) The set $\{U \subseteq \mathbb{R} \mid U \text{ is infinite}\} \cup \{\varnothing\}$ is a topology on $\mathbb{R}$.

   False. (Hint: Consider finite intersections.)

   (e) Consider $[0, 2]$ as a subspace of $\mathbb{R}$ with the standard topology. The subset $(1, 2] \subseteq [0, 2]$ is open.

   True. (Hint: What is $(1, 3) \cap [0, 2]$?)

   (f) Consider $[0, 2]$ as a subspace of $\mathbb{R}$ with the standard topology. Then $(0, 1) \cup \{2\} \subseteq [0, 2]$ is open.

   False. (Hint: What must be true of an open subset of $\mathbb{R}$ containing 2?)

   (g) Let $f : X \to \mathbb{R}$ be a map from a space X to $\mathbb{R}$ (with the standard topology). Then X is continuous if and only if $f^{-1}(B_r(x))$ is open for every **rational** numbers $x, r \in \mathbb{Q}$, $r > 0$.

   True. (Hint: Check that $\{B_r(x) \mid r, x \in \mathbb{Q}\}$ is a basis for the standard topology.)

   (h) Let X be a space with the property that points $\{x\}$ are **open**. Then X is a T_1-space.

   True. (Hint: Check that X in fact must have the discrete topology.)

   (i) The Cartesian product of two quotient maps is a quotient map.

   False. (Hint: Homework 8, Problem #5.)

(j) The composite of two quotient maps is a quotient map.

True. (Hint: Check that the three conditions are preserved by composition.)

(k) The set $\{a, b, c, d\}$ with the topology $\{\varnothing, \{c\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}\}$ is connected.

True. (Hint: Check that no proper nonempty subset is both open and closed.)

2. Each of the following statements is either true or false. If the statement holds in general, write "`True`". Otherwise, state a counterexample. **No justification necessary.**

Note: You can get partial credit for correctly writing "`False`" without a counterexample.

Remark: These solutions have more justification than needed. On the exam it is enough to simply state the counterexample.

(i) Let $f : X \to Y$ be a function of sets X and Y. If there is a function g so that $g \circ f : X \to X$ is the identity map of X, then f is invertible.

False. For example, consider the case where $X = \mathbb{R}$, $Y = \mathbb{R} \times \mathbb{R}$, f is the inclusion of the x-axis, and g is the projection onto the x-axis, as follows.

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \times \mathbb{R} \\ x &\longmapsto (x, 0) \end{aligned} \qquad\qquad \begin{aligned} g : \mathbb{R} \times \mathbb{R} &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto x \end{aligned}$$

Then $g \circ f$ is the identity on $\mathbb{R}$, but f (and g) are not invertible.

(ii) Let X be a topological space, and $S \subseteq X$ be a subset with no limit points. Then S is closed.

True.

(iii) Let X be a topological space, and $C \subseteq X$ a closed set. The inclusion map $C \to X$ is a closed map.

True.

(iv) Let A be a subset of a metric space (X, d). Then any element of ∂A must be both a limit point of A, and a limit point of $X \setminus A$.

False. For example, consider $X = \mathbb{R}$ with the standard topology, and $A = \{1\}$. Then $\partial A = \{1\}$ but 1 is not a limit point of A.

(v) Let X be a topological space, and $S \subseteq X$. Then $\partial S = \partial\left(\overline{S}\right)$.

False. Consider $S = \mathbb{Q} \subseteq \mathbb{R}$ with the Euclidean metric. Then $\partial S = \mathbb{R}$ but $\partial(\overline{S}) = \partial\mathbb{R} = \varnothing$.

(vi) Let X be a topological space, and $S \subseteq X$. Then $\overline{S} = X \setminus \mathrm{Int}(X \setminus S)$.

True.

(vii) If $A \subseteq B$, then $\mathrm{Int}(A) \subseteq \mathrm{Int}(B)$.

True.

(viii) If $A \subseteq B$, then all limits points of A are also limit points of B.

True.

(ix) If $\mathrm{Int}(A) = \mathrm{Int}(B)$ and $\overline{A} = \overline{B}$, then $A = B$.

False. Consider $A = [0, 1) \subseteq \mathbb{R}$ and $B = (0, 1] \subseteq \mathbb{R}$. Then

$$\mathrm{Int}(A) = \mathrm{Int}(B) = (0, 1) \qquad \text{and} \qquad \overline{A} = \overline{B} = [0, 1]$$

but $A \neq B$.

(x) If $\mathrm{Int}(A) = \overline{A}$, then A is both open and closed.

True.

(xi) Let $(X, \mathcal{T})$ be a topological space, and let x, y be **distinct** points in X. Let $(a_n)_{n\in\mathbb{N}}$ be the sequence $x\ y\ x\ y\ x\ y\ x\ y \cdots$. Then $(a_n)_{n\in\mathbb{N}}$ does not converge.

False. For example, consider $X = \{x, y\}$ with the indiscrete topology. Then the sequence converges to both x and to y.

(xii) If a sequence of points $(a_n)_{n\in\mathbb{N}}$ in a topological space X converges to a point a_∞, then a_∞ is a limit point of the set $\{a_n \mid n \in \mathbb{N}\}$.

False. For example, consider the constant sequence $(0)_{n\in\mathbb{N}}$ in $\mathbb{R}$. Then the sequence converges to 0, but 0 is not a limit point of the set $\{0\} \subseteq \mathbb{R}$.

(xiii) Suppose $(a_n)_{n\in\mathbb{N}}$ is a sequence in a topological space, and that x is a limit point of the set $\{a_n \mid n \in \mathbb{N}\}$. Then there is some subsequence converging to x.

False. Let $X = \mathbb{N}$ with the topology $\{\varnothing\} \cup \{U \mid 1 \in U\}$, and consider the sequence $a_n = n$. Then 2 is a limit point of the sequence, since every neighbourhood of 2 contains $a_1 = 1$. However, there is no subsequence converging to 2, since (for example) the neighbourhood $\{1, 2\}$ of 2 contains only finitely many terms in the sequence.

(xiv) There is no sequence in $\mathbb{R}$ (with the standard topology) with the property that, for any $r \in \mathbb{R}$, there is some subsequence converging to $\mathbb{R}$.

False. For example, consider any sequence $(a_n)_{n\in\mathbb{N}}$ that enumerates the (countable set) $\mathbb{Q}$. Then $(a_n)_{n\in\mathbb{N}}$ has a subsequence of rational numbers that converge to any real number.

(xv) Let X and Y be metric spaces, and $f : X \to Y$ a continuous function. If $B \subseteq X$ is bounded, then $f(B)$ is bounded.

False. For example, consider $X = Y = (0, \infty)$ in the Euclidean metric, and the continuous map $f : (0, \infty) \to (0, \infty)$ given by $f(x) = \frac{1}{x}$. Then $(0, 1) \subseteq (0, \infty)$ is bounded, but $f\Big((0, 1)\Big) = (1, \infty)$ is not bounded.

(xvi) Let $f : X \to Y$ be continuous function. If X is Hausdorff, then $f(X)$ is Hausdorff.

False. For example, consider the identity map

$$\begin{aligned} I : (\mathbb{R}, \text{discrete topology}) &\longrightarrow (\mathbb{R}, \text{indiscrete topology}) \\ x &\longrightarrow x \end{aligned}$$

Then I is continuous and $(\mathbb{R}, \text{discrete topology})$ is Hausdorff, but $\text{im}(f) = (\mathbb{R}, \text{indiscrete topology})$ is not Hausdorff.

(xvii) Let $f : X \to Y$ be a continuous map. Then the restriction of f to any subspace of X is continuous (with respect to the subspace topology).

True.

(xviii) Let $(f_n : X \to \mathbb{R})_{n\in\mathbb{N}}$ be a sequence of continuous functions that converge pointwise to a function $f : X \to \mathbb{R}$. If f is continuous, then the functions must converge uniformly.

False. For example, consider the functions

$$\begin{aligned} f_n &: (0, 1) \to \mathbb{R} \\ f_n(x) &= x^n. \end{aligned}$$

Then the sequence converges to the constant function $f(x) = 0$, but convergence is not uniform.

(xix) An injective quotient map is necessarily a homeomorphism.

True.

(xx) Let $p : X \to A$ be a quotient map. If X is a T_1-space, then A is a T_1-space.

False. Consider the map

$$f : \mathbb{R} \longrightarrow \{a, b\}$$
$$f(x) = \begin{cases} a, & x \in (-\infty, 0) \\ b, & x \in [0, \infty) \end{cases}$$

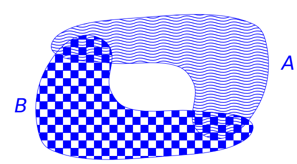
Then $\mathbb{R}$ is a T_1-space, but the quotient topology $\Big\{\varnothing, \{a\}, \{a, b\}\Big\}$ on the codomain does not have the T_1 property, since $\{a\}$ is not closed.

(xxi) If $(X, \mathcal{T})$ is a connected topological space, and $\mathcal{T}'$ is a coarser topology on X , then $(X, \mathcal{T}')$ is also connected.

True.

(xxii) Let A and B be nonempty subsets of a topological space $(X, \mathcal{T})$. If A and B are connected and $A \cap B$ is nonempty, then $A \cap B$ is connected.

False. For example, consider the following two subsets of $\mathbb{R}^2$ (with the Euclidean metric).



3. Let X be a set, and suppose that $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on X.
   (a) Show that the intersection $\mathcal{T}_1 \cap \mathcal{T}_2$ is a topology on X.

   Solution. Suppose $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on X. To check that $\mathcal{T}_1 \cap \mathcal{T}_2$ is a topology, we must check the three conditions.
   First, we observe that $\varnothing, X$ must be contained in both $\mathcal{T}_1$ and $\mathcal{T}_2$ (since $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies), hence $\varnothing, X \in \mathcal{T}_1 \cap \mathcal{T}_2$.
   Next, we supppose that $U_1, U_2 \in \mathcal{T}_1 \cap \mathcal{T}_2$. Then $U_1, U_2 \in \mathcal{T}_1$, so $U_1 \cap U_2 \in \mathcal{T}_1$ by definition of a topology. The same argument shows $U_1 \cap U_2 \in \mathcal{T}_2$. We conclude that $U_1 \cap U_2 \in \mathcal{T}_1 \cap \mathcal{T}_2$.
   Finally, suppose $\{U_i\}_{i \in I}$ is a collection of open subsets in $\mathcal{T}_1 \cap \mathcal{T}_2$. Then $\{U_i\}_{i \in I} \subseteq \mathcal{T}_1$, so $\bigcup_{i \in I} U_i \in \mathcal{T}_1$. Similarly, $\bigcup_{i \in I} U_i \in \mathcal{T}_2$. Hence $\bigcup_{i \in I} U_i \in \mathcal{T}_1 \cap \mathcal{T}_2$.
   We conclude that $\mathcal{T}_1 \cap \mathcal{T}_2$ is a topology on X.

   (b) Show by example that the union $\mathcal{T}_1 \cup \mathcal{T}_2$ need not be a topology on X.

   Solution. Consider the two topologies on the set $\{a, b, c\}$,

$$\mathcal{T}_1 = \{\varnothing, \{a\}, \{a, b, c\}\} \qquad \text{and} \qquad \mathcal{T}_2 = \{\varnothing, \{b\}, \{a, b, c\}\}.$$

   Their union

$$\mathcal{T}_1 \cup \mathcal{T}_2 = \{\varnothing, \{a\}, \{b\}, \{a, b, c\}\}$$

   is not a topology, because it does not contain the union $\{a\} \cup \{b\} = \{a, b\}$.

4. Let $(X, \mathcal{T})$ be a topological space, and let $S \subseteq X$ be a subset. Show that $\partial(\partial S) = \partial S$ if and only if ∂S has empty interior.

Solution. First suppose that $x \in \operatorname{Int}(\partial S)$. Then $x \in \partial S$. But, the boundary and interior of any set are disjoint, so $x \notin \partial(\partial S)$. In this case we conclude that $\partial(\partial S) \neq \partial S$.

Next suppose that ∂S has empty interior. Recall that we showed, for a general set A, that $\partial A = \overline{A} \setminus \operatorname{Int}(A)$. Then

$$\begin{aligned}
\partial(\partial S) &= \overline{\partial S} \setminus \operatorname{Int}(\partial S) \\
&= \overline{\partial S} \setminus \varnothing && \text{(by assumption)} \\
&= \overline{\partial S} \\
&= \partial S && \text{(since boundaries are closed)}
\end{aligned}$$

as claimed.

5. Let $(X, \mathcal{T})$ be a topological space, and let $S \subseteq X$ be a subset. Suppose that the subspace topology on S is the discrete topology. Prove or give a counterexample: S is closed as a subset of X.

Solution. The statement is false. Consider, for example, the $X = \mathbb{R}$ with the standard topology, and the subset

$$S = \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{N} \right\}.$$

Then S has the discrete topology. To check this, it suffices to show that each point $\{\frac{1}{n}\} \subseteq S$ is open in the subspace topology. Since any nonempty subset of S is a union of its points, it will follow that arbitrary subsets of S are open. But

$$\left\{ \frac{1}{n} \right\} = S \cap \left(\frac{1}{n+1}, \frac{1}{n-1} \right)$$

is the intersection of S and an open subset, and is therefore open. Hence S has the discrete topology.

The set S, however, is not closed, since 0 is a limit point of S that is not contained in S.

6. Consider the following functions $f : \mathbb{R} \to \mathbb{R}$.

- $f(x) = x$
- $f(x) = x^2$
- $f(x) = x + 1$
- $f(x) = 0$
- $f(x) = \cos(x)$
- $f(x) = -x$

Determine whether these functions are continuous ...

(a) ... when $\mathbb{R}$ has the topology $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\varnothing\} \cup \{\mathbb{R}\}$.

(b) ... when $\mathbb{R}$ has the cofinite topology.

(c) ... when $\mathbb{R}$ has the cocountable topology.

(d) ... when $\mathbb{R}$ has the topology $\mathcal{T} = \{\mathbb{R}\} \cup \{U \subseteq \mathbb{R} \mid 0 \notin U\}$.

(e) ... when $\mathbb{R}$ has the topology $\mathcal{T} = \{\varnothing\} \cup \{U \subseteq \mathbb{R} \mid 0 \in U\}$.

Solution. The identity function $f(x) = x$ and the constant function $f(x) = 0$ are continuous with respect to any topology on $\mathbb{R}$. We will give full solutions in the case $f(x) = x + 1$.

- $\mathbb{R}$ has the topology $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\varnothing\} \cup \{\mathbb{R}\}$.
  In this case, f is continuous. The preimages of the open sets are always open:
  $$f^{-1}(\varnothing) = \varnothing, \qquad f^{-1}(\mathbb{R}) = \mathbb{R}, \qquad f^{-1}((a, \infty)) = (a - 1, \infty).$$
- $\mathbb{R}$ has the cofinite topology.
  In this case, f is continuous. It suffices to check that the preimages of closed subsets are closed. But the proper closed subsets of $\mathbb{R}$ are exactly the finite subsets, and (because f is one-to-one) the preimage of any finite subset is finite.
- $\mathbb{R}$ has the cocountable topology.
  In this case, f is continuous. Again it suffices to check that the preimages of closed subsets are closed. But the proper closed subsets of $\mathbb{R}$ are exactly the countable subsets, and (because f is one-to-one) the preimage of any countable subset is countable.
- $\mathbb{R}$ has the topology $\mathcal{T} = \{\mathbb{R}\} \cup \{U \subseteq \mathbb{R} \mid 0 \notin U\}$.
  In this case, f is not continuous. The preimage of the open set $\{1\}$ is the not-open set $\{0\}$.
- $\mathbb{R}$ has the topology $\mathcal{T} = \{\varnothing\} \cup \{U \subseteq \mathbb{R} \mid 0 \in U\}$.
  In this case, f is not continuous. The preimage of the open set $\{0\}$ is the not-open set $\{-1\}$.

7. Consider the set $X = \{a, b, c, d\}$ with the topology
$$\mathcal{T} = \{\varnothing, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}.$$

(a) Write down a permutation of the elements of X that is not continuous.

Solution. Consider the permutation σ given by
$$a \mapsto b \mapsto a, \quad c \mapsto c, \quad d \mapsto d.$$
Then $\{a\}$ is open, but $\sigma^{-1}(\{a\}) = \{b\}$ is not open, so σ is not continuous.

(b) Is there a nonconstant, nonidenty map $X \to X$ that is continuous?

Solution. Consider the map $f : X \to X$ given by
$$a, b \mapsto a, \quad c \mapsto c, \quad d \mapsto d.$$
Then
$$\begin{aligned} &f^{-1}(\varnothing) = \varnothing && f^{-1}(\{a\}) = \{a, b\} \\ &f^{-1}(\{a, b\}) = \{a, b\} && f^{-1}(\{a, c\}) = \{a, b, c\} \\ &f^{-1}(\{a, b, c\}) = \{a, b, c\} && f^{-1}(\{a, b, d\}) = \{a, b, d\} \\ &f^{-1}(\{a, b, c, d\}) = \{a, b, c, d\} && \end{aligned}$$
and so we see that f is continuous.

(c) Is there a non-identity permutation of the elements of X that is a homeomorphism?

Solution. There is no non-identity permutation of X that is a homeomorphism. Recall that, if f is homeomorphism, then f must be a bijection, and the induced maps

$$\begin{aligned} \mathcal{T} &\longrightarrow \mathcal{T} \\ A &\longmapsto f(A) \\ B &\longmapsto f^{-1}(B) \end{aligned}$$

must be bijections. Note that this means if A is an n-element set, then $f(A)$ and $f^{-1}(A)$ must also be n-element sets. Note further that (for example), since the element d is contained in one 3-element and one 4-element open set (and no other open set), f must map d to an element that is contained in one 3-element and one 4-element open set (and no other open set). But d is the only such element, so $f(d) = d$. Observe more generally that:

- a is the only element for which $\{a\}$ is open
- b is the only element contained in exactly 4 open sets
- c is the only element contained in exactly 3 open sets
- d is the only element conttained in exactly 2 open sets.

Any homeomorphism must therefore map every one of these elements to themselves.

(d) Let $[0, 1]$ be the closed interval with the standard topology. Is there a non-constant continuous surjective map $[0, 1] \to X$?

Solution. Consider the map

$$f : [0,1] \to X$$
$$f(x) = \begin{cases} a, & x \in [0, \frac{1}{2}) \\ b, & x \in [\frac{1}{2}, 1]. \end{cases}$$

Then we can verify that the preimage of each open set is open:

$$\begin{aligned} &f^{-1}(\varnothing) = \varnothing \\ &f^{-1}(\{a\}) = f^{-1}(\{a, c\}) = \left[0, \frac{1}{2}\right) \\ &f^{-1}(\{a, b\}) =^{-1} (\{a, b, c\}) = f^{-1}(\{a, b, d\}) = f^{-1}(\{a, b, c, d\}) = [0, 1] \end{aligned}$$

and so conclude that f is continuous.

(e) Is there a non-constant continuous map $X \to [0, 1]$?

Solution. There is no such continuous map. Suppose (for the sake of contradiction) that f were such a map. Then the image of f is a finite set containing at least two elements (since f is non-constant). The image must therefore be discrete. If $x, y \in f(X)$ with $x \neq y$, then $f^{-1}(\{x\}$ and $f^{-1}(\{y\}$ must both be disjoint open subsets of X. However, at most one of these sets can contain a, and every open subset of X contains a, a contradiction.

8. Prove the following propositions.

(a) A function $f : X \to Y$ of topological space is *continuous at the point* $x \in X$ if for every neighbourhood U of $f(x)$, there is a neighbourhood V of x such that $f(V) \subseteq U$.

Proposition. A function $f : X \to Y$ of topological spaces is continuous if and only if it is continuous at every point $x \in X$.

(b) **Proposition.** Let $f : X \to Y$ be a function of topological spaces. Then f is continuous if and only if $f^{-1}(\text{Int}(B)) \subseteq \text{Int}(f^{-1}(B))$ for every $B \subseteq Y$.

Solution. First, suppose that f is a function satisfying the condition stated in the Proposition. We will show that f is continuous. Let $U \subset Y$ be an open subset; we must show that $f^{-1}(U)$ is open. But

$$\begin{aligned}
\text{Int}(f^{-1}(U)) &\subseteq f^{-1}(U) && \text{(by definition of interior)},\\
&= f^{-1}(\text{Int}(U)) && \text{(since } U \text{ is open)},\\
&\subseteq \text{Int}(f^{-1}(U)) && \text{(by assumption)}.
\end{aligned}$$

This sequence of set-containments is only possible if we have equality at each step, and in particular we conclude that $\text{Int}(f^{-1}(U)) = f^{-1}(U)$, and it follows that $f^{-1}(U)$ is open.

Now suppose that f is continuous, and we will verify that f satisfies the condition in the proposition. For any set $B \subseteq Y$, the interior $\text{Int}(B)$ is open, so (by definition of continuity) $f^{-1}(\text{Int}(B))$ is open. Moreover, since $\text{Int}(B) \subseteq B$, it follows that $f^{-1}(\text{Int}(B)) \subseteq f^{-1}(B)$. But, we proved that every open subset of a set A must be contained in $\text{Int}(A)$. In this case, because $f^{-1}(\text{Int}(B))$ is an open subset $f^{-1}(B)$, it must be contained in $\text{Int}(f^{-1}(B))$. This completes the proof.

(c) **Proposition.** Let $f : X \to Y$ be a function of topological spaces, and let $\mathcal{S}$ be a subbasis for Y. Then f is continuous if and only if $f^{-1}(U)$ is open for every subbasis element $U \subseteq \mathcal{S}$.

9. Let $\{0, 1\}$ be a topological space with the discrete topology. Let X be a topological space and $A \subseteq X$ a subset. Define the *characteristic function on* A

$$\chi_A : X \longrightarrow \{0, 1\}$$
$$\chi_A(x) = \begin{cases} 1, x \in A \\ 0, x \notin A \end{cases}$$

(a) Prove that χ_A is continuous at a point $x \in X$ if and only if $x \notin \partial A$.

(b) Prove that χ_A is continuous if and only if A is both open and closed.

10. You proved on Homework #6 that if $X = A \cup B$ for closed sets A and B, and $f : X \to Y$ is a function whose restriction to A and B are both continuous, then f is continuous. Prove or disprove that the same statement holds if X is a union of (possibly infinitely many) closed sets A_i, with $f|_{A_i}$ continuous for each i.

Solution. The statement is false. Consider any discontinuous function $f : \mathbb{R} \to \mathbb{R}$, for example,

$$f(x) = \begin{cases} 1, & x \in [0, 1], \\ 0, & x \notin [0, 1]. \end{cases}$$

Since points $\{x\} \subseteq \mathbb{R}$ are closed, we can write $\mathbb{R}$ as the union of closed sets $\mathbb{R} = \bigcup_{x \in \mathbb{R}} \{x\}$. But we can verify that the restriction of f to $\{x\}$ is continuous: the preimage of any subset of $\mathbb{R}$ must be equal to one of the two subsets $\varnothing$ and $\{x\}$ of $\{x\}$, and both subsets are open. Hence f is a discontinuous function with the property that its restriction to every closed subset is continuous.

11. Let (X, d) be a metric space, and let $\mathcal{B}$ be a basis for the topology $\mathcal{T}_d$ induced by d.
    (a) Let $S \subseteq X$ be a subset, and $s \in S$. Show that s is an interior point of S if and only if there is some element $B \in \mathcal{B}$ such that $s \in B$ and $B \subseteq S$.
    (b) Deduce that $\operatorname{Int}(S) = \bigcup_{B \in \mathcal{B}, B \subseteq S} B$.

12. Let $f : X \to Y$ be a function of topological spaces. Prove that f is open if and only if $f(\operatorname{Int}(A)) \subseteq \operatorname{Int}(f(A))$ for all sets $A \subseteq X$.

13. (a) Let $(X, \mathcal{T}_X)$ be a Hausdorff topological space, and let $x_1, \ldots, x_n$ be a finite collection of points in X. Show that there are open sets $U_1, \ldots, U_n$ such that $x_i \in U_i$, and which are pairwise disjoint (this means $U_i \cap U_j = \varnothing$ for all $i \neq j$).

    Solution. Let $x_1, \ldots, x_n$ be a finite collection of points in X. Since X is Hausdorff, for every distinct pair of points x_i and x_j, there exist disjoint neighbourhoods $U_{i,j}$ of x_i and $U_{j,i}$ of x_j. For each $i = 1, \ldots, n$, define the set

    $$U_i = \bigcap_{\substack{j \in \{1,2,\ldots,n\} \\ j \neq i}} U_{i,j}.$$

    Since $i \in U_{i,j}$ for all j, their intersection U_i is a neighbourhood of i. We claim that these neighbourhoods are pairwise dijoint. Suppose that $x \in U_i \cap U_j$. But then $x \in U_i \subseteq U_{i,j}$ and $x \in U_j \subseteq U_{j,i}$, which is a contradiction since $U_{i,j}$ and $U_{j,i}$ were disjoint by construction. Thus we have constructed pairwise disjoint sets $U_1, \ldots, U_n$ with $x_i \in U_i$ for each i.

    (b) Let X be a finite topological space. Prove if X is Hausdorff, then it has the discrete topology.

    Solution. Since X is finite, we can denote its elements by $x_1, \ldots, x_n$ for $n = |X|$. By part (a), there are neighbourhoods U_i if x_i for each i that are pairwise disjoint. Fix i. Since U_i is disjoint from U_j for each $j \neq i$, we infer that $x_j \notin U_i$ for all $j \neq i$, so $U_i = \{x_i\}$. Thus we deduce that points in X are open. Since every nonempty set can be expressed as a union of its points, we conclude that every subset of X is open. Hence X has the discrete topology.

14. Let $A \subseteq \mathbb{R}$ be a nonempty bounded subset. Prove that its supremum $\sup(A)$ is either in the set A, or it is a limit point of A.

15. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of points in a topological space X that converges to $a_\infty \in X$.
    (a) Prove that, if we replace the topology on X with any coarser topology, then the sequence $(a_n)_{n \in \mathbb{N}}$ will still converge to a_∞.
    (b) Show by example that, if we replace the topology on X with any finer topology, then the sequence $(a_n)_{n \in \mathbb{N}}$ may no longer converge to a_∞.

16. Let X be a metric space, and let $a_\infty \in X$. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence in X with the property that every subsequence of $(a_n)_{n \in \mathbb{N}}$ has a subsequence that converges to a_∞. Prove that $(a_n)_{n \in \mathbb{N}}$ converges to a_∞.

    Solution: We will prove the contrapositive: We will show that if $(a_n)_{n \in \mathbb{N}}$ does not converge to a_∞, then we can construct a subsequence $(a_{n_i})_{i \in \mathbb{N}}$ with the property that none of its (sub)subsequences converges to a_∞.

    Recall that a sequence $(a_n)_{n \in \mathbb{N}}$ converges to a_∞ if, for every $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $a_n \in B_\epsilon(a_\infty)$ for all $n > N$. This means that the sequence $(a_n)_{n \in \mathbb{N}}$ *fails* to converge to a_∞ if for some $\epsilon > 0$, for every $N \in \mathbb{N}$ there is some $n > N$ so that $a_n \notin B_\epsilon(a_\infty)$.

Assume that $(a_n)_{n\in\mathbb{N}}$ does not converge, and choose such a value of $\epsilon > 0$. It follows that there is some $n_1 \in \mathbb{N}$ so that $a_{n_1} \notin B_\epsilon(a_\infty)$. But then, taking $N_1 = n_1$, there is necessarily some $n_2 > n_1$ so that $a_{n_2} \notin B_\epsilon(a_\infty)$. And, again, taking $N_2 = n_2$, there is some $n_3 > n_2$ such that $a_{n_3} \notin B_\epsilon(a_\infty)$. Continuing with this procedure, by induction, we obtain a subsequence $(a_{n_i})_{i\in\mathbb{N}}$ that has no terms contained in $B_\epsilon(a_\infty)$.

We will show that no subsequence of $(a_{n_i})_{i\in\mathbb{N}}$ converges to a_∞. Let $(a_{n_{i_j}})_{j\in\mathbb{N}}$ be any (sub)subsequence of this subsequence. Consider the value $\epsilon > 0$ as defined above. If $(a_{n_{i_j}})_{j\in\mathbb{N}}$ converged to a_∞, then $a_{n_{i_j}}$ must be contained in $B_\epsilon(a_\infty)$ for infinitely many values of j. However, the terms $a_{n_{i_j}}$ by construction are not contained in $B_\epsilon(a_\infty)$ for *any* values of j. Thus, the subsequence does not converge to a_∞.

We conclude that a sequence $(a_n)_{n\in\mathbb{N}}$ must converge to a_∞ if it has the property that each of its subsequences has a (sub)subsequence converging to a_∞.

17. Let X be a topological space $A \subseteq X$, and $h : X \to \mathbb{R}$ a continuous function. Suppose there is a constant c such that $h(x) \leq c$ for all $x \in A$. Prove that $h(x) \leq c$ for all $x \in \overline{A}$.

Solution. Since h is a continuous map, we know that $f^{-1}(C)$ is closed for every closed subset $C \subseteq \mathbb{R}$. In particular, $f^{-1}((-\infty, c])$ is closed. By assumption, $A \subseteq f^{-1}((-\infty, c])$. By definition $\overline{A}$ is the intersection of every closed set containing A, hence $\overline{A}$ is contained in every closed set containing A. Notably, $\overline{A} \subseteq f^{-1}((-\infty, c])$. Therefore $f(x) \leq c$ for all $x \in \overline{A}$.

18. Suppose that X and Y are nonempty topological spaces, and that the product topology on $X \times Y$ is Hausdorff. Prove that X and Y are both Hausdorff.

Solution. We first claim that, if a space Z is Hausdorff, then so is any subspace $S \subseteq Z$. Consider two distinct points $x, y \in S$. Since Z is Hausdorff, there are open neighbourhoods U_x and U_y of x and y, respectively, that are disjoint. Then (by definition of the subspace topology) $U_x \cap S$ and $U_y \cap S$ are open subsets of S. They are necessarily still neighbourhoods of x and y, respectively, and still must be disjoint. Thus S is Hausdorff.

Fix $x_0 \in X$ and $y_0 \in Y$; these elements exist by the assumption that X and Y are nonempty. Next we claim that the maps

$$\begin{aligned} i_X : X &\to X \times Y \\ x &\mapsto (x, y_0) \end{aligned} \qquad\qquad \begin{aligned} i_Y : Y &\to X \times Y \\ y &\mapsto (x_0, y) \end{aligned}$$

are embeddings. It is clear that they are injective, and so bijections onto their images. We will verify that i_X is a embedding by checking that, if we restrict its codomain to its image, it is both continuous and open. A similar argument shows that i_Y is an embedding.

Let U be an open set in X. Then

$$i_X(U) = U \times \{y_0\} = (U \times Y) \cap (X \times \{y_0\}) = (U \times Y) \cap \mathrm{im}(i_X).$$

Since $U \times Y$ is open in the product topology, we conclude that $i_X(U)$ is open in the subspace topology on $\mathrm{im}(i_X)$. By Homework 6 Problem #3, because the identity map $X \times Y \to X \times Y$ is continuous, it is continuous in each variable, which shows in particular that i_X is continuous.

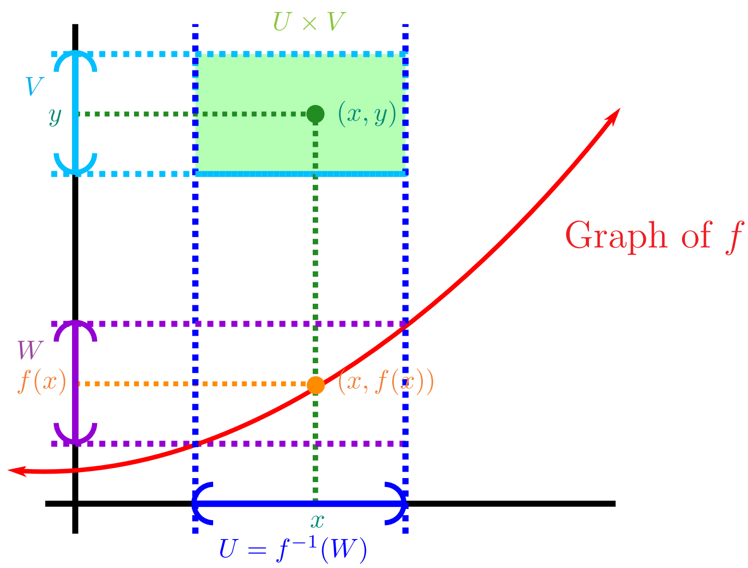
Now, since $X \times \{y_0\}$ is a subspace of the Hausdorff space $X \times Y$, we deduce that it is Hausdorff. We have proved moreover that this subspace is homeomorphic to X. Since the Hausdorff property is a homeomorphism invariant, we conclude that X is Hausdorff. A similar argument shows that Y is Hausdorff.

19. (a) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $f : X \to Y$ be a function. Recall that the *graph* of f is defined to be the subset of $X \times Y$

$$\{ \ (x, f(x)) \in X \times Y \mid x \in X \ \}.$$

Suppose that Y is Hausdorff. Show that, if f is continuous, then the graph of f is a closed subset of $X \times Y$ with respect to the subspace topology $\mathcal{T}_{X \times Y}$.

Solution: Denote the graph by $G \subseteq X \times Y$. To show that G is closed, we wish to show that its complement is open. To do this, it suffices to show that any point (x, y) in the complement of G has an open neighbourhood that is contained in the complement of G.
Let (x, y) be a point in the complement of G. Then $y \neq f(x)$. Since Y is Hausdorff, we can therefore find disjoint open neighbourhoods V of y and W of $f(x)$ in Y. Since f is continuous, the subset $U = f^{-1}(W)$ is open in X. Because $f(x) \in W$, we know x is contained in its preimage U.



We claim that the subset $U \times V$ is an open neighbourhood of (x, y) in $X \times Y$ contained in the complement of G. The subset $U \times V$ contains (x, y) by construction, and it is open in $X \times Y$ by the definition of the product topology. So it suffces to show that $U \times V$ is contained in the complement of G.
Consider $(u, v) \in U \times V$. Then $f(u) \in W$ by definition of U. But $v \in V$ and V is disjoint from W by construction, so $v \neq f(u)$. We conclude that $(u, v) \notin G$, and that $U \times V$ is contained in the complement of G as claimed.

(b) Show by example that, if Y is not Hausdorff, the graph of a function $f : X \to Y$ need not be closed.

Solution: Consider the set $\mathbb{R}$ with the indiscrete topology, and consider the identity function $I : \mathbb{R} \to \mathbb{R}$. Then the graph of I is the diagonal $\Delta = \{(x, x) \mid x \in \mathbb{R}\} \subseteq \mathbb{R} \times \mathbb{R}$. The product topology on $\mathbb{R} \times \mathbb{R}$ is generated by the basis

$$\varnothing \times \varnothing = \varnothing, \quad \mathbb{R} \times \varnothing = \varnothing, \quad \varnothing \times \mathbb{R} = \varnothing, \quad \mathbb{R} \times \mathbb{R}$$

and hence is the indiscrete topology on $\mathbb{R}$. Since Δ is a proper nonempty subset of $\mathbb{R} \times \mathbb{R}$, it is not closed.

20. Let $f, g : X \to Y$ be continuous maps between topological spaces. Show that if Y is Hausdorff, then the set $S = \{x \in X \mid f(x) = g(x)\}$ is closed.

21. (a) Let $S \subseteq X$ be a subset of a topological space X. Explain why, to show that S is closed, it suffices to show that S is the preimage of a closed set under a continuous function.

Solution. We proved that a function $f : X \to Y$ of topological spaces is continuous if and only if $f^{-1}(C) \subseteq X$ is closed for every closed set $C \subseteq Y$. Hence, if a subset of $A \subseteq X$ is the preimage of a closed set C under a continuous function f, then A must be closed.

(b) Show that the following subsets of $\mathbb{R}^2$ (with the usual topology) are closed:

$$\{(x, y) \mid xy = 1\} \qquad S^1 = \{(x, y) \mid x^2 + y^2 = 1\} \qquad D^2 = \{(x, y) \mid x^2 + y^2 \leq 1\}$$

Solution. It is a result from real analysis that a polynomial $p(x, y)$ in variables x and y defines a continuous function from $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Then we can use part (a) to check that each of the above sets is closed:

- $\{(x, y) \mid xy = 1\}$ is the preimage of the closed set $\{1\} \subseteq \mathbb{R}$ under the continuous function $p(x, y) = xy$.
- $S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$ is the preimage of the closed set $\{1\} \subseteq \mathbb{R}$ under the continuous function $q(x, y) = x^2 + y^2$.
- $D^2 = \{(x, y) \mid x^2 + y^2 \leq 1\}$ is the preimage of the closed set $(-\infty, 1] \subseteq \mathbb{R}$ under the continuous function $q(x, y) = x^2 + y^2$.

22. Let X be a topological space.

(a) Suppose that X is Hausdorff. Let $x \in X$. Show that the intersection of all open sets containing x is equal to $\{x\}$.

Solution. Fix $x \in X$. Let

$$U = \bigcap_{\substack{x \in U_x \\ U_x \text{ open}}} U_x.$$

Since $x \in U_x$ for all x, it follows that $x \in U$. We need to show that, under the assumption that x is Hausdorff, $y \notin U$ for all $y \neq x$. But if $y \neq x$, then there are disjoint neighbourhoods U_x of x and U_y of y. In paritcular, $y \notin U_x$. Hence $y \notin U$, and we conclude that $U = \{x\}$ as claimed.

(b) Show that the converse statement does not hold. Specifically, suppose $(X, \mathcal{T})$ is a infinite set with the cofinite topology. Show that $(X, \mathcal{T})$ is not Hausdorff, but for any $x \in X$ the intersection of all open sets containing X is equal to $\{x\}$.

Solution. Consider the set $X = \mathbb{R}$ with the cofinite topology. Then X is not Hausdorff, as any two nonempty open sets have an uncountable intersection. But we will show that, for any $x \in X$,

$$\bigcap_{\substack{x \in U_x \\ U_x \text{ open}}} U_x = x.$$

Fix x in X. Again, it is clear that $x \in U$. To show equality, we must show that $y \notin U$ for all $y \neq x$. But given $y \neq x$, the sett $U_x = \mathbb{R} \setminus \{y\}$ is an open neighbourhood of x, and so $U \subseteq \mathbb{R} \setminus \{y\}$. We conclude that $y \notin U$, and therefore that $U = \{x\}$.

23. Let $\mathbb{R}^\omega = \prod_{\mathbb{N}} \mathbb{R}$ (so an element of $\mathbb{R}^\omega$ is precisely a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers.) Fix sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ with $a_n \neq 0$ for all n. Define

$$\begin{aligned} h : \mathbb{R}^\omega &\longrightarrow \mathbb{R}^\omega \\ (x_1, x_2, x_3, \dots) &\longmapsto (a_1x_1 + b_1, a_2x_2 + b_2, a_3x_3 + b_3, \dots) \end{aligned}$$

Determine whether h is a homeomorphism if its domain and codomain are given the product topology, and if they are given the box topology.

Hint: h is a homeomorphism in both cases. See Homework #7 Warm-up Problem 3.

24. Find an explicit homeomorphism bewteen the intervals $(0, 1)$ and $(1, \infty)$ with the Euclidean metric.

25. Let (X, d) be a metric space. Show that the function $D(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}$ defines a new metric on X.

Solution.

First note that, since $d(x, y) \geq 0$ for all x, y, the denominator of $D(x, y)$ is always strictly positive (and never zero), so D is well-defined. To chekc that it si a metric, we will check the three axioms.

- (Positivity). Since $d(x, y) \geq 0$ for all $x, y \in X$, the function $D(x, y)$ is the ratio of a nonnegative number to a strictly positive number, and is therefore always nonnegative. Moreover, $D(x, y) = 0$ if and only if its numerator $d(x, y) = 0$, which happens if and only if $x = y$ since d is a metric.
- (Symmetry). $D(y, x) = \dfrac{d(y, x)}{1 + d(y, x)} = \dfrac{d(x, y)}{1 + d(x, y)} = D(x, y)$ for all $x, y \in X$ since d is a metric and therefore symmetric in x and y.
- (Triangle inequality). First observe that the function $F(t) = \frac{t}{1+t}$ has positive derivative $F'(t) = \frac{1}{(1+t)^2}$, and therefore is increasing in t. Then for any $x, y, z \in X$, we know $d(x, z) \leq d(x, y) + d(y, z)$, and therefore $F(d(x, z)) \leq F(d(x, y) + d(y, z))$. Expanding this inequality,

$$\begin{aligned} D(x, z) &= \frac{d(x, z)}{1 + d(x, z)} \\ &= F(d(x, y) \\ &\leq F(d(x, y) + d(y, z)) \\ &= \frac{d(x, y) + d(y, z)}{1 + d(x, y) + d(y, z)} \\ &= \frac{d(x, y)}{1 + d(x, y) + d(y, z)} + \frac{d(y, z)}{1 + d(x, y) + d(y, z)} \\ &\leq \frac{d(x, y)}{1 + d(x, y)} + \frac{d(y, z)}{1 + d(y, z)} \\ &= D(x, y) + D(y, z). \end{aligned}$$

26. Let $f_n : X \to Y$ be a sequence of functions from a topological space X to a metric space (Y, d). Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in X converging to x_∞.

   (a) Show that, if the sequence $\big(f_n\big)_{n\in\mathbb{N}}$ converges uniformly to a function f_∞, then the sequence $\big(f_n(x_n)\big)_{n\in\mathbb{N}}$ converges to $f_\infty(x_\infty)$.

   Solution. To verify that $\big(f_n(x_n)\big)_{n\in\mathbb{N}}$ converges to $f_\infty(x_\infty)$, we must show that, for every $\epsilon > 0$, there is some $N \in \mathbb{N}$ so that $f_n(x_n) \in B_\epsilon\Big(f_\infty(x_\infty)\Big)$ for all $n \geq N$.

   So fix $\epsilon > 0$. Because the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly, there exists some $N_1 \in \mathbb{N}$ so that $d(f_n(x), f_\infty(x)) < \frac{\epsilon}{2}$ for all $n \geq N_1$ and all $x \in X$. In particular, taking $x = x_n$,

$$d(f_n(x_n), f_\infty(x_n)) < \frac{\epsilon}{2} \qquad \text{for all } n \geq N_1.$$

   Next, recall that, because f_∞ is the uniform limit of a sequence of continuous functions, by Homework #8 Problem 3(a), the function f_∞ must also be continuous. Hence, the preimage $f_\infty^{-1}(B)$ of the ball $B = B_{\frac{\epsilon}{2}}\Big(f_\infty(x_\infty)\Big)$ under f_∞ is an open subset of X containing x_∞. Then, because the sequence $(x_n)_{n\in\mathbb{N}}$ converges to x_∞, there must be some $N_2 \in \mathbb{N}$ so that $x_n \in f_\infty^{-1}(B)$ for all $n \geq N_2$. In other words, $f(x_n) \in B = B_{\frac{\epsilon}{2}}\Big(f_\infty(x_\infty)\Big)$ for all $n \geq N_2$. This means that

$$d(f_\infty(x_n), f_\infty(x_\infty)) < \frac{\epsilon}{2} \qquad \text{for all } n \geq N_2.$$

   So let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, we find that

$$\begin{aligned} d(f_n(x_n), f_\infty(x_\infty)) &\leq d(f_n(x_n), f_\infty(x_n)) + d(f_\infty(x_n), f_\infty(x_\infty)) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

   Hence for all $n \geq N$, $f_n(x_n) \in B_\epsilon\Big(f_\infty(x_\infty)\Big)$. We conclude that $\big(f_n(x_n)\big)_{n\in\mathbb{N}}$ converges to $f_\infty(x_\infty)$, as claimed.

   (b) Show that this conclusion need not hold if the sequence of functions $\big(f_n\big)_{n\in\mathbb{N}}$ only converges pointwise to f_∞.

   Solution. Recall from Homework 8 Problem #3(b) that the sequence of functions

$$\begin{aligned} f_n : [0,1] &\to \mathbb{R} \\ x &\mapsto x^n \end{aligned}$$

   converges pointwise to the function

$$f(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1. \end{cases}$$

   Consider the sequence $(1 - \frac{1}{n})_{n\in\mathbb{N}}$ in $\mathbb{R}$. Then

$$f_n\left(1 - \frac{1}{n}\right) = \left(1 - \frac{1}{n}\right)^n \xrightarrow{n\to\infty} \frac{1}{e},$$

   whereas

$$f\left(\lim_{n\to\infty}\left(1 - \frac{1}{n}\right)\right) = f(1) = 1.$$

27. Consider the sequence of functions $f_n : [0.1] \to [0,1]$ defined by the equations $f_n(x) = \begin{cases} nx, & 0 \le x \le \frac{1}{n} \\ 1, & \frac{1}{n} \le x \le 1. \end{cases}$
    (a) Show that this sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise to the constant function $f(x) = 1$.
    (b) Show that this sequence does not converge uniformly.
    (c) Conclude that, even when a sequence of continuous functions converges pointwise to a continuous function, the convergence need not be uniform.

28. Let (X, d) be a metric space. Fix $x_0 \in X$ and $r > 0$ in $\mathbb{R}$.
    (a) Show that the closure of the open ball $B_r(x_0)$ is contained in $\{x \mid d(x_0, x) \le r\}$.

    Solution. By part (d), the set $\{x \mid d(x_0, x) \le r\}$ is closed, and (by inspection) it contains $B_r(x_0)$. By the definition of closure, it therefore must contain the closure of $B_r(x_0)$.

    (b) Give an example of a metric space where the closure is always equal to this set.

    Solution. A result from real analysis is that, over $\mathbb{R}$ (with the standard topology),

$$\overline{B_r(x_0)} = \{x \mid d(x_0, x) \le r\}.$$

    (c) Give an example of a metric space X and a ball $B_r(x_0)$ whose closure is a strict subset of $\{x \mid d(x_0, x) \le r\}$.

    Solution. Recall that the discrete metric on $\mathbb{R}$ is given by

$$d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \ne y \end{cases}$$

    Consider the ball $B_1(0) = \{0\}$. This set is both open and closed, so it is equal to its own closure. In contrast, $\{x \mid d(0, x) \le 1\} = \mathbb{R}$.

    (d) Show that the set $\{x \mid d(x_0, x) \le r\}$ is closed in any metric space.

    Solution. Let B denote the set $\{x \mid d(x_0, x) \le r\}$. To show that B is closed, we will show that its complement is open, which we will do by proving that an arbitrary point y in the complement is an interior point of the complement. So suppose that $y \notin B$, this means that $d(x_0, y) > r$. Let $\epsilon = d(x_0, y) - r$; it follows that $\epsilon > 0$. We will show that the ball $B_\epsilon(y)$ is contained in the complement of B, which will prove that y is an interior point and thus conclude the proof. Let $z \in B_\epsilon(y)$; our goal is to show that $z \notin B$. But

$$\begin{aligned} d(x_0, y) &\le d(x_0, z) + d(z, y) && \text{(by the triangle inequality)} \\ d(x_0, y) &< d(x_0, z) + \epsilon && \text{(since } z \in B_\epsilon(y)\text{)} \\ d(x_0, y) &< d(x_0, z) + \Big(d(y, x_0) - r\Big) \\ r &< d(x_0, z) \end{aligned}$$

Thus, $z \notin B$. This shows that $B_\epsilon(y)$ is a neighbourhood of y contained in the complement of B. Then y is an interior point of the complement; the complement is open, and B is closed.

29. Let (X,d) be a metric space, and let $S \subseteq X$ be a **finite** subset of X. Prove that S ...

    (a) is closed. (b) is bounded. (c) has no limit points.

    (d) Show that S may or may not have empty interior $\mathrm{Int}(S) = \varnothing$.

30. Given an index set I, define metrics on $\mathbb{R}^I$ as follows. For points $\mathbf{x} = (x_i)_{i\in I}$ and $\mathbf{y} = (y_i)_{i\in I}$, recall that we defined the uniform metric by

$$d_U(\mathbf{x},\mathbf{y}) = \sup_{i\in I}\Big\{ \min\{|x_i - y_i|, 1\}\Big\}.$$

Moreover, when $I = \mathbb{N}$, define a new metric let

$$d(\mathbf{x},\mathbf{y}) = \sup_{i\in \mathbb{N}}\left\{ \frac{\min\{|x_i - y_i|, 1\}}{i}\right\}.$$

   (a) Verify that the uniform metric d_U is in fact a metric. We proved that this metric induces the uniform topology on $\mathbb{R}^I$.

   (b) Suppose $I = \mathbb{N}$. Verify that d is a metric.

   (c) Show that d induces the product topology on $\mathbb{R}^\omega$.

Solution: The following proof is from Munkres Theorem 20.5.

Recall that the product topology on $\mathbb{R}^\omega$ is, by definition, generated by the basis

$$\left\{ \prod_{i\in\mathbb{N}} U_i \;\middle|\; U_i \subseteq \mathbb{R} \text{ is open; there is some } N\in\mathbb{N} \text{ such that } U_i = \mathbb{R} \text{ for all } i \geq N \right\}.$$

We first show that the product topology is finer than the metric topology. Choose an open set U in the metric topology. To check that U is open in the product topology, we will check that an arbitrary point $x = (x_i)_{i\in\mathbb{N}}$ is an interior point of U in the product topology. Choose $\epsilon > 0$ so that $B^d_\epsilon(x) \subseteq U$. Then choose N large enough that $\frac{1}{N} < \epsilon$. Let

$$V = (x_1 - \epsilon, x_1+\epsilon) \times (x_2-\epsilon, x_2+\epsilon) \times \cdots \times (x_N - \epsilon, x_N + \epsilon) \times \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\cdots$$

By construction, $x \subseteq V$, and V is open in the product topology. If we can show that $V \subseteq B^d_\epsilon(x) \subseteq U$, then we can conclude that x is an interior point of U, and U is open in the product topology. So let $y = (y_i)_{i\in\mathbb{N}} \in V$. Then

$$\begin{aligned}
&d(x,y)\\
&= \sup\left\{ \frac{\min\{|x_1-y_1|,1\}}{1}, \frac{\min\{|x_2-y_2|,1\}}{2}, \ldots, \frac{\min\{|x_N-y_N|,1\}}{N}, \frac{\min\{|x_{N+1}-y_{N+1}|,1\}}{N+1}, \ldots, \right\}\\
&\leq \sup\left\{ \frac{\min\{|x_1-y_1|,1\}}{1}, \frac{\min\{|x_2-y_2|,1\}}{2}, \ldots, \frac{\min\{|x_N-y_N|,1\}}{N}, \frac{1}{N+1}, \frac{1}{N+2}, \ldots, \right\}\\
&\leq \sup\left\{ \frac{\epsilon}{1}, \frac{\epsilon}{2}, \ldots, \frac{\epsilon}{N}, \frac{1}{N+1}, \frac{1}{N+2}, \ldots, \right\} \qquad \text{(since } y \in V)\\
&\leq \max\left\{\epsilon, \frac{1}{N+1}\right\}\\
&\leq \epsilon \qquad \text{(by choice of } N).
\end{aligned}$$

Thus $y \in B^d_\epsilon(x)$. We conclude that $V \subseteq B^d_\epsilon(x)$, so the metric topology is contained in the product topology.

Next, we check that the metric topology is finer than the product topology. Let W be a basis element $W = \prod_{i \in \mathbb{N}} W_i$ for the product topology; it suffices to show that W is open in the metric topology. Let $w = (w_i)_{i \in I} \in W$; we wish to find a ball $B^d_\delta(w)$ contained in W. This ball will show that w is an interior point of W in the metric topology, and thus that W is open in the metric topology.

Let $N \in \mathbb{N}$ be such that $W_i = \mathbb{R}$ for all $i \geq \mathbb{N}$. Since W_i is open in $\mathbb{R}$ for all i, for each i there is some $\delta_i > 0$ so that the interval $(w_i - \delta_i, w_i + \delta_i) \subseteq W_i$. Moreover, (possibly by making each interval even smaller), we can choose each δ_i so that $\delta_i < 1$. Let

$$\delta = \min\left\{\frac{\delta_1}{1}, \frac{\delta_2}{2}, \ldots, \frac{\delta_N}{N}\right\},$$

and note that since N is finite, $\delta > 0$. We claim that $B^d_\delta(w) \subseteq W$.

So let $z = (z_i)_{i \in I} \in B^d_\delta(w)$. Then for each $i \in \mathbb{N}$,

$$\begin{aligned}
\frac{\min\{|z_i - w_i|, 1\}}{i} &\leq \sup_{i \in \mathbb{N}} \left\{\frac{\min\{|z_i - w_i|, 1\}}{i}\right\} && \text{(by definition of sup)} \\
&= d(z, w) && \text{(by definition of } d\text{)} \\
&< \delta && \text{(since } z \in B^d_\delta(w)\text{)} \\
&= \min\left\{\frac{\delta_1}{1}, \frac{\delta_2}{2}, \ldots, \frac{\delta_N}{N}\right\} && \text{(by definition of } \delta\text{)} \\
&\leq \frac{\delta_j}{j} \qquad \text{for every } j = 1, \ldots, N, && \text{(by definition of min).}
\end{aligned}$$

In particular, for $i = 1, \ldots, N$,

$$\frac{\min\{|z_i - w_i|, 1\}}{i} < \frac{\delta_i}{i}$$
$$\min\{|z_i - w_i|, 1\} < \delta_i.$$

Our assumption that $\delta_i < 1$ ensures that the $\min\{|z_i - w_i|, 1\}$ must in fact equal $|z_i - w_i|$, and so we deduce that $|z_i - w_i| < \delta_i$ for each $i = 1, \ldots, N$, so $z_i \in (w_i - \delta_i, w_i + \delta_i)$. Hence $z \in W$, and we conclude that W is open in the metric topology. This concludes the proof.

31. Let (X, d) be a metric space. Show that the map $d : X \times X \to \mathbb{R}$ is continuous with respect to the product topology on X and the standard topology on $\mathbb{R}$. Show moreover that the topology on X induced by d is the coarsest topology making d continuous.

32. Show that metrizability is a topological property. In other words, show that, if X and Y are homeomorphic topological spaces, then X is metrizable if and only if Y is.

    Hint: Suppose that d is a metric on Y, and $f : X \to Y$ is a homeomorphism. Show that

    $$D(x_1, x_2) = d(f(x_1), f(x_2))$$

    defines a metric on X.

33. Let X be a finite set (of, say, n elements), and let d be a metric on X. What is the topology $\mathcal{T}_d$ on X induced by d? Show in particular that this topology will be the same for every possible metric d.

34. We proved in class that, if d and $\tilde{d}$ are two metrics on a set X, then d and $\tilde{d}$ are topologically equivalent if they satisfy the following condition: For each $x \in X$, there exist positive constants $\alpha, \beta > 0$ such that for every $y \in X$,
$$\alpha\, d(x,y) \leq \tilde{d}(x,y) \leq \beta\, d(x,y).$$
(Note that α and β depend on x but are independent of y.) Show that the converse of this statement fails: find a set X and equivalent metrics d and $\tilde{d}$ on X that fail this condition.

Solution. Consider the following two metrics on $\mathbb{N}$: the metric d_E induced by the Euclidean metric, and the discrete metric d_D. Both of these metrics induce the discrete topology on $\mathbb{N}$, but they fail the above condition. For example, fix $x = 1 \in \mathbb{N}$. Then $d_D(1,n) \leq 1$ for all $n \in \mathbb{N}$, but $d_E(1,n) = (n-1)$ grows without bound as n increases. The condition would require a constant $\alpha > 0$ such that
$$\alpha\,(n-1) = \alpha\, d_E(1,n) \leq d_D(1,n) \leq 1$$
for all $n \in \mathbb{N}$. But these inequalities fail if we choose $n > 1 + \frac{1}{\alpha}$, and we conclude that no such α exists.

35. Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose that $f : X \to Y$ is a function that preserves distances in the sense that
$$d_Y\Big(f(x_1), f(x_2)\Big) = d_X(x_1, x_2) \qquad \text{for all } x_1, x_2 \in X.$$
Show that f is continuous, and is an embedding of topological spaces. Such maps are called *isometric embeddings*.

36. Let J be an uncountable index set, and consider the product $\mathbb{R}^J$ with the product topology associated to the standard topology on $\mathbb{R}$. In this question, we will show that this product is Hausdorff but not metrizable. Define tthe subset $A = \{(x_j)_{j\in J} \mid x_j = 1 \text{ for all by finitely many } j \in J\ \}$ of $\mathbb{R}^J$. Let $\mathbf{0}$ denote the element $(0)_{j\in J} \in \mathbb{R}^J$ that is constant 0 in every component.

(a) Consider an arbitrary basis element $U = \prod_{j\in J} U_j$ for the product topology on $\mathbb{R}$ with $\mathbf{0} \in U$. Explain why U contains an element of A. Conclude that $\mathbf{0} \in \overline{A}$.

(b) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points $a_n = (a_{n,j})_{j\in J} \in A$. Explain why there is an index $j \in J$ such taht $a_{n,j} = 1$ for every n. *Hint:* J is uncountable.

(c) Construct a neighourhood $V = \prod_{j\in J} V_j$ of $\mathbf{0}$ that does not contain a_n for any $n \in \mathbb{N}$. Deduce that the sequence $(a_n)_{n\in\mathbb{N}}$ does not converge to $\mathbf{0}$.

(d) Conclude that the product $\mathbb{R}^J$ is Hausdorff, but not metrizable.

37. Let X be a topological space, and X^* a quotient space of X. Show that X^* is a T_1–space if and only if every equivalence class in X^* is closed as a subset of X.

Solution. Let $p : X \to X^*$ be the quotient map. We proved on Homework 6 #1(b) that $X*$ is a T_1–space if and only if $\{x\}$ is closed for every $x \in X^*$. But, we proved on Homework 8 #4(a) that, under a quotient map, a set $\{x\}$ is closed if and only if $p^{-1}(\{x\}) \subseteq X$ is closed. But the subsets $p^{-1}(\{x\}) \subseteq X$ are precisely the equivalence classes defining X^*, which concludes the proof.

38. Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the projection onto the first factor. Show that the restriction of π_1 to the subset
$$\{(x,y) \mid xy = 1\} \cup \{(0,0)\}$$
is continuous and surjective, but is not a quotient map.

Solution. Note that the set $\{(x, y) \mid xy = 1\}$ is precisely the graph of the function $\frac{1}{x}$ on $\mathbb{R} \setminus \{0\}$.

Since π_1 is continuous, its restriction to any subspace is also continuous with respect to the subspace topology.

To verify that the restriction of π_1 surjects, we note that $(0, 0) \in \pi_1^{-1}(\{0\})$, and $(x, \frac{1}{x}) \in \pi_1^{-1}(\{x\})$ for all $x \neq 0$.

Finally, we will check that the restriction of π_1 is not a quotient map. The subset $\{(x, y) \mid xy = 1\}$ is closed in $\mathbb{R}^2$, since it is the preimage of the closed set $\{1\}$ under the continuous function

$$\begin{aligned} \mathbb{R}^2 &\to \mathbb{R} \\ (x, y) &\mapsto xy. \end{aligned}$$

However, the image of $\{(x, y) \mid xy = 1\}$ under π_1 is the subset $\mathbb{R} \setminus \{0\}$ of $\mathbb{R}$, which is not closed in $\mathbb{R}$. Thus π_1 is not a quotient map.

39. Let $p : X \to Y$ be a quotient map of topological spaces, and $A \subseteq X$ a subset. Show that, if p is an open map, then the restriction of p to A is also a quotient map.

40. Let $X \subseteq \mathbb{R}^2$ be the subspace $\{(x, n) \mid n \in \mathbb{N}, x \in [0, 1]\}$ consisting of the horizontal line $[0, 1] \times \{n\}$ for each natural number n. Let $Y \subseteq \mathbb{R}^2$ be the subspace $\{(x, \frac{x}{n}) \mid n \in \mathbb{N}, x \in [0, 1]\}$ consisting of the line of slope $\frac{1}{n}$ through the origin for each natural number n. Define a map $g : X \to Y$ by $g(x, n) = (x, \frac{x}{n})$.
    (a) Verify that g is continuous and surjective.
    (b) Determine whether g is a quotient map.

    Hint: The map g is not a quoient map. Consider the set of points $\{(\frac{1}{n}, n) \mid n \in \mathbb{N}\}$ in X, and its image $\{(\frac{1}{n}, \frac{1}{n^2}) \mid n \in \mathbb{N}\}$ in Y. Show that this set is closed in X, but its image has limit point $(0, 0)$ and therefore is not closed.

41. Let X be a topological space and $Y \subseteq X$. Show that a separation of Y is precisely a pair of disjoint nonempty sets $A, B \subseteq Y$ whose union is Y, such that neither set contains a limit point of the other.

Solution. Recall from Homework #4 Problem 8 that the closure of a set C is the union of C and its limit points. In particular, a set C is closed if and only if it contains all of its limit points.

Now, suppose that A and B are a separation of Y. Since A is closed in the subspace topology, it contains all of its limit points in the subspace Y. Moreover, since A and B are disjoint and $B \subseteq Y$, B cannot contain any limit points of A. Similarly A cannot containa any limit points of B.

Conversely, suppose that A and B are disjoint nonempty subsets of Y whose union is Y, such that neither set contains a limit point of the other. Since $B = Y \setminus A$ does not contain any limit points of A, we infer that A contains all of its limit points. Therefore, A is closed, and $B = Y \setminus A$ is open. The same argument, reversing the roles of A and B, shows that A is open. We can therefore conclude that A and B separate Y.

42. Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of connected subsets of a space X. Suppose that $A_n \cap A_{n+1} \neq \varnothing$ for each n. Show that the union $\bigcup_n A_n$ is connected.

43. Let $(X, \mathcal{T})$ be a topological space. Let $\{A_n \mid n \in \mathbb{N}\}$ be a family of connected subspaces in X such that $A_{n+1} \subseteq A_n$ for every $n \in \mathbb{N}$. Is $\bigcap_{n \in \mathbb{N}} A_n$ is necessarily connected?

Solution. This statement is false. For example, let $X = \mathbb{R}$ with the cofinite topology. Consider the family of subsets

$$A_n = [0, 1/n) \cup [1, 1 + 1/n).$$

We will first verify that the set A_n is connected. Suppose that $A_n = U \cup V$ were a separation of A_n. But then the instersection $U \cap V$ must be the complement of a finite set, so $(U \cap V) \cap A_n \neq \varnothing$. This contradicts the premise that U and V are a separation of A_n, and we conclude that A_n is connected.

Next, observe that

$$A = \bigcap_{n \in \mathbb{N}} A_n = \{0, 1\},$$

and we will verify that A is disconnected. Note that $\mathbb{R} \setminus \{1\}$ is open in X, and $(\mathbb{R} \setminus \{1\}) \cap A = \{0\}$, so $\{0\}$ is open in the subspace topology on A. Similarly $\{1\}$ is open in A. But then $A = \{0\} \cup \{1\}$ is a separation of A, and A is disconnected.

44. Let $(X, \mathcal{T})$ be a topological space, and let $A, B \subseteq X$. Suppose $A \cup B$ and $A \cap B$ are connected. Prove that if A and B are both closed or both open, then A and B are connected.

We will give a solution in the case that A and B are both open.

Lemma. If S is a subspace of a topological space Z, and $S \subseteq Z$ is open, then $U \subseteq S$ is open in the subspace S if and only if U is open in Z.

Proof. A set U is open in S if and only if $U = V \cap S$ for some open subset $V \subseteq Z$. Since the intersection of two open sets is open, it follows that U is open in Z. Conversely, if $U \subseteq S$ is open in Z, then $U = U \cap S$ is open in S.

Solution. We will show that A is connected; the same argument will show that B is connected. Suppose that $A = C \cup D$ is a separation of A. Since $A \cap B$ is connected, from the lemma in class it must be contained entirely in C or entirely in D. Say WLOG that $A \cap B \subseteq D$. Now we claim that the sets $C' = C$ and $D' = D \cup B$ separate $A \cup B$. We must check four conditions:

- C' and D' are nonempty.
  This result follows since $C \subset C'$ and $D \subset D'$ are nonempty by assumption that they separate A.
- C' and D' are disjoint.
  C does not intersect D, by assumption that C and D separate A. Moreover, C does not intersection B, since $C \subseteq A$ and all points in the intersection $A \cap B$ lie in D.
- C' and D' are open.
  C is open in A, and therefore by the lemma above it is open in $A \cup B$. Similarly, D is open in $A \cup B$. Then, since B is open, $D \cup B$ is open.
- $(A \cup B) = (C' \cup D')$.
  This follows since $C' \cup D' = C \cup (D \cup B) = (C \cup D) \cup B = A \cup B$.

So C' and D' separate $A \cup B$. This contradicts the premise that $A \cup B$ is connected, and we conclude that A is connected.

45. **Definition (Adherent sets).** Let $(X, \mathcal{T}_X)$ be a topological space, and let $A, B \subseteq X$. Then A and B are called *adherent* if

$$(A \cap \overline{B}) \cup (\overline{A} \cap B) \neq \varnothing.$$

(a) Give examples of disjoint adherent subsets of $\mathbb{R}$ (with the Euclidean metric).

(b) Let $(X, \mathcal{T}_X)$ be a topological space and $A, B, C \subseteq X$. Prove or give a counterexample: if A and B are adherent, and B and C are adherent, then A and C are adherent.

(c) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $f : X \to Y$ be a continuous map. Prove that, if A and B are adherent subsets of X, then $f(A)$ and $f(B)$ are adherent subsets of Y.

(d) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be Hausdorff topological spaces. Suppose that $f : X \to Y$ has the property that, whenever A and B are adherent subsets of X, then $f(A)$ and $f(B)$ are adherent subsets of Y. Prove that f is continuous.

Solution. Recall from Homework #5 Problem 2(a) that a function $f : X \to Y$ is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$.

So suppose that $f : X \to Y$ has the property that, whenever A and B are adherent subsets of X, then $f(A)$ and $f(B)$ are adherent subsets of Y. Let $A \subseteq X$. Our goal is to show that $f(\overline{A}) \subseteq \overline{f(A)}$. Let $x \in \overline{A}$; we wish to show $f(x) \in \overline{f(A)}$. Observe that $\{x\}$ and A are adherent sets, since

$$x \quad \in \quad (\overline{A} \cap \{x\}) \quad \subseteq \quad (A \cap \overline{\{x\}}) \cup (\overline{A} \cap \{x\}).$$

Then, by assumption, $f(\{x\}) = \{f(x)\}$ and $f(A)$ are adherent sets. This means that

$$(f(A) \cap \overline{\{f(x)\}}) \cup (\overline{f(A)} \cap \{f(x)\}) \neq \varnothing.$$

But, since Y is Hausdorff, the point $\{f(x)\}$ is closed, and hence $\{f(x)\} = \overline{\{f(x)\}}$. So

$$(f(A) \cap \{f(x)\}) \cup (\overline{f(A)} \cap \{f(x)\}) \neq \varnothing.$$

This implies that $f(x) \subseteq \overline{f(A)}$. Hence $f(\overline{A}) \subseteq \overline{f(A)}$, and we conclude that f is continuous.