Final Exam

Math 590
29 April 2019
Jenny Wilson

Name: ____

Instructions: This exam has 4 questions for a total of 50 points.

The exam is closed-book. No books, notes, cell phones, calculators, or other devices are permitted. Scratch paper is available.

Fully justify your answers unless otherwise instructed. You may quote any results proved in class or on the homeworks without proof, but please include a precise statement of the result you are quoting.

You have 120 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

Question	Points	Score
1	20	
2	20	
3	6	
4	4	
Total:	50	

1. (20 points) Each of the following statements is either true or false. If the statement holds in general, write "`True`". Otherwise, write "`False`". **No justification necessary.**

   (a) Consider $\mathbb{N}$ with the standard topology. Let $S = \{(a_n)_{n\in\mathbb{N}} \mid a_n \in \mathbb{N}, \lim_{n\to\infty} a_n = 0\}$ be the set of all sequences converging to zero. Then S is countable.

   (b) The set $\{U \subseteq \mathbb{R} \mid U \text{ is finite}\} \cup \{\mathbb{R}\}$ is a topology on $\mathbb{R}$.

   (c) Consider $\mathbb{Q}$ as a subspace of $\mathbb{R}$ with the standard topology. The subset $\{0\}$ is open.

   (d) Let X be a finite topological space. If X is a T_1-space, then X is discrete.

   (e) Let A be a subset of a topological space T. Then the set of limit points of A is closed.

   (f) If X is a T_1-space, then limits of sequences in X are unique.

   (g) Let $\mathcal{T}$ be a topology on a set X. If $\mathcal{T}$ has the T_1 property, then $\mathcal{T}$ is finer than the cofinite topology on X.

(h) For spaces X and Y, let $A \subseteq X$ and $B \subseteq Y$ be closed sets. They $A \times B$ is a closed subset of $X \times Y$ in the product topology.

(i) Let X and Y be topological spaces and consider $X \times Y$ with the product topology. If $U \subseteq X \times Y$ is open, then $U = U_X \times U_Y$ for some open sets $U_X \subseteq X$ and $U_Y \subseteq Y$.

(j) Let $\{X_i\}_{i \in I}$ and $\{Y_i\}_{i \in I}$ be families of nonempty topological spaces, and $f_i : X_i \to Y_i$ a function for each i. Suppose that $\prod_{i \in I} X_i$ and $\prod_{i \in I} Y_i$ are both given the product topology, or both given the box topology. Then the function

$$\begin{aligned} \prod f_i : \prod_{i \in I} X_i &\longmapsto \prod_{i \in I} Y_i \\ (x_i)_{i \in I} &\longmapsto \big(f_i(x_i)\big)_{i \in I} \end{aligned}$$

is continuous if and only if f_i is continuous for each i.

(k) Let $C \subseteq \mathbb{R}^\omega$ be the set of all sequences $(a_n)_{n \in \mathbb{N}} \in \mathbb{R}^\omega$ that are eventually zero. Then C is closed in the uniform topology.

(l) If a subspace A of a space X is connected, then ∂A is connected.

(m) For any index set J, the product $(0,1)^J$ in the product topology is connected.

(n) Let $p : X \to Y$ be a quotient map. If Y is a T_1-space, then so is X.

(o) Consider $X = \mathbb{R}$ with the discrete topology. Then its one-point compactification $\hat{X} = X \cup \{\infty\}$ has the topology $\{U \subseteq \hat{X} \mid \infty \notin U\} \cup \{\hat{X}\}$.

(p) Let X be a locally compact Hausdorff space, and let $\hat{X}$ be its one-point compactification. Let f be a continuous map from X to a Hausdorff space Y. Then there is at most one value $\hat{f}(\infty) \in Y$ that would extend the function f to a continuous function $\hat{f} : \hat{X} \to Y$.

(q) The cofinite topology on $\mathbb{R}$ is separable but not second countable.

(r) The space $\mathbb{N}$ with the cofinite topology is a Baire space.

(s) Let X be a space. Then $\mathscr{C}(X, \mathbb{R})$ in the uniform topology is a Baire space.

(t) The set of functions $\{f_n(x) = nx \mid n \in \mathbb{N}\}$ in $\mathscr{C}([0, 1], \mathbb{R})$ is equicontinuous.

2. (20 points) Each of the following statements is either true or false. If the statement holds in general, write "`True`". Otherwise, state a counterexample. **No justification necessary.**

   (i) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.

   (ii) If $A \subseteq B$, then $\partial A \subseteq \partial B$.

   (iii) Let $f : X \to Y$ be a map of topological spaces. If f is continuous, then $\overline{f(A)} \subseteq f(\overline{A})$ for every $A \subseteq X$.

   (iv) Let S be a subset of a topological space X, and suppose that the subspace topology on S is the discrete topology. Then S is closed.

   (v) Let $\{X_i\}_{i \in I}$ be a collection of metric spaces with the metric topology. Then their product $\prod_{i \in I} X_i$ is metrizable in the box topology.

   (vi) Let A be a path-connected subspace of a space X. Then $\overline{A}$ is path-connected.

   (vii) If X is a connected topological space, then so are all its quotient spaces.

(viii) Let $f : X \to Y$ be an open map of topological spaces. Then the restriction of f to any subspace of X is open (with respect to the subspace topology).

(ix) Let $f : X \to Y$ be a continuous invertible map, and suppose that on every connected component of X the map f restricts to a homeomorphism to a connected component of Y. Then f is a homeomorphism.

(x) There does not exist a path from a to d in the space $X = \{a, b, c, d\}$ with the topology $\Big\{\varnothing, \{a\}, \{d\}, \{a, d\}, \{c, d\}, \{a, c, d\}, X\Big\}$.

(xi) Any compact space X is second countable.

(xii) Any second countable space X is compact.

(xiii) If X is complete with respect to a metric d, then X is complete with respect to any metric equivalent to d.

(xiv) If a space X is compact, then X is limit point compact.

(xv) The continuous image of a regular space is regular.

(xvi) Every discrete space is normal.

(xvii) A subspace of a separable space is separable.

(xviii) For a topological space X and a metric space Y the set of all continuous, bounded functions in Y^X is a closed subset of Y^X in the uniform topology.

(xix) If $(f_n)_{n\in\mathbb{N}}$ is a convergent sequence of bounded continuous functions $\mathbb{R} \to \mathbb{R}$ in the compact-open topology, then the limit f is bounded.

(xx) Consider the set $\mathscr{F} = \{f : [0,1] \to \mathbb{R} \mid f(x) = ax^2 + bx + c \text{ with } a, b, c \in [0,1]\}$ of continuous functions. Then any sequence of functions in $\mathscr{F}$ has a subsequence that converges uniformly to some continuous function $[0,1] \to \mathbb{R}$.

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3. (a) (2 points) For each of the following sequences of elements of $\mathbb{R}^\omega$, circle all the topologies on $\mathbb{R}^\omega$ with respect to which the sequence converges. **No justification needed.**

(i)

$$\begin{aligned} \mathbf{a_1} &= (1,1,1,1,1,\ldots) \\ \mathbf{a_2} &= (0,2,2,2,2,\ldots) \\ \mathbf{a_3} &= (0,0,3,3,3,\ldots) \\ \mathbf{a_4} &= (0,0,0,4,4,\ldots) \\ &\vdots \end{aligned}$$

`product topology`

`uniform topology`

`box topology`

(ii)

$$\begin{aligned} \mathbf{b_1} &= (1,1,1,1,1,\ldots) \\ \mathbf{b_2} &= (0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\ldots) \\ \mathbf{b_3} &= (0,0,\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},\ldots) \\ \mathbf{b_4} &= (0,0,0,\tfrac{1}{4},\tfrac{1}{4},\ldots) \\ &\vdots \end{aligned}$$

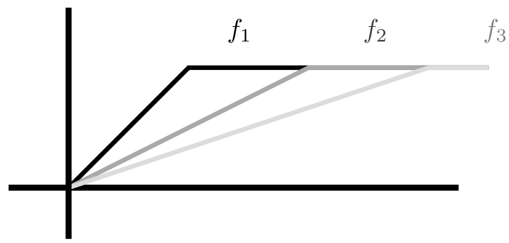
`product topology`

`uniform topology`

`box topology`

(b) (2 points) For each of the sequences of continuous functions $f_n : [0,\infty) \to \mathbb{R}$, circle all the topologies on $\mathscr{C}([0,\infty),\mathbb{R})$ with respect to which the sequence converges.

(i) $f_n(x) = \begin{cases} \frac{x}{n}, & x \in [0,n] \\ 1, & x > n \end{cases}$

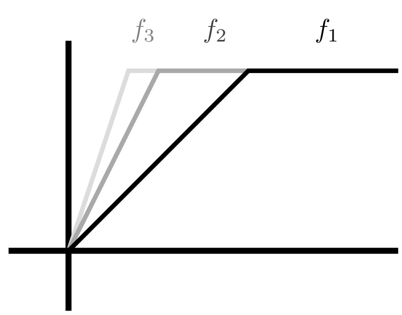


`topology of pointwise convergence`

`compact-open topology`

`uniform topology`

(ii) $f_n(x) = \begin{cases} nx, & x \in [0,\frac{1}{n}] \\ 1, & x > \frac{1}{n} \end{cases}$



`topology of pointwise convergence`

`compact-open topology`

`uniform topology`

(c) (2 points) Circle all terms that apply. The topology $\{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\mathbb{R}\} \cup \{\varnothing\}$ on $\mathbb{R}$ is ...

`compact` `path-connected` `separable` T_4 `(normal)`

4. (4 points) Let X and Y be topological spaces. We saw in class that, in general, the projection map $\pi_X : X \times Y \to X$ may not be closed. Show that, if Y is compact, then π_X is a closed map.

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