Final Exam

Math 590
29 April 2019
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Name: ______________________________

Instructions: This exam has 4 questions for a total of 50 points.

The exam is closed-book. No books, notes, cell phones, calculators, or other devices are permitted. Scratch paper is available.

Fully justify your answers unless otherwise instructed. You may quote any results proved in class or on the homeworks without proof, but please include a precise statement of the result you are quoting.

You have 120 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

Question	Points	Score
1	20	
2	20	
3	6	
4	4	
Total:	50	

1. (20 points) Each of the following statements is either true or false. If the statement holds in general, write "`True`". Otherwise, write "`False`". **No justification necessary.**

   (a) Consider $\mathbb{N}$ with the standard topology. Let $S = \{(a_n)_{n\in\mathbb{N}} \mid a_n \in \mathbb{N}, \lim_{n\to\infty} a_n = 0\}$ be the set of all sequences converging to zero. Then S is countable.

   True. *Hint:* Every such sequence is eventually zero. This set of sequences can be written as a countable union of countable sets.

   (b) The set $\{U \subseteq \mathbb{R} \mid U \text{ is finite}\} \cup \{\mathbb{R}\}$ is a topology on $\mathbb{R}$.

   False. *Hint:* Topologies are closed under arbitrary unions.

   (c) Consider $\mathbb{Q}$ as a subspace of $\mathbb{R}$ with the standard topology. The subset $\{0\}$ is open.

   False. *Hint:* The open subsets of $\mathbb{Q}$ are generated by the basis $(a, b) \cap \mathbb{Q}$ for open intervals $(a, b) \subset \mathbb{R}$.

   (d) Let X be a finite topological space. If X is a T_1-space, then X is discrete.

   True. *Hint:* Every subset of X is a finite union of points, and therefore closed.

   (e) Let A be a subset of a topological space T. Then the set of limit points of A is closed.

   False. *Hint:* This is true for a T_1-space (Homework #6 Problem 1(e)) but not in general. See for example Quiz 3 Problem 1(b).

   (f) If X is a T_1-space, then limits of sequences in X are unique.

   False. *Hint:* This is true of Hausdorff spaces but not T_1-spaces in general. Consider $\mathbb{R}$ with cofinite topology. Every real number is a limit of the sequence $(n)_{n\in\mathbb{N}}$.

   (g) Let $\mathcal{T}$ be a topology on a set X. If $\mathcal{T}$ has the T_1 property, then $\mathcal{T}$ is finer than the cofinite topology on X.

   True. *Hint:* If X is a T_1-space, then points (hence finite unions of points) are closed, so cofinite subsets of X are open.

(h) For spaces X and Y, let $A \subseteq X$ and $B \subseteq Y$ be closed sets. They $A \times B$ is a closed subset of $X \times Y$ in the product topology.

True. *Hint:* See Homwork 7 Problem 1(c).

(i) Let X and Y be topological spaces and consider $X \times Y$ with the product topology. If $U \subseteq X \times Y$ is open, then $U = U_X \times U_Y$ for some open sets $U_X \subseteq X$ and $U_Y \subseteq Y$.

False. *Hint:* The product topology is generated by open subsets of the form $U_X \times U_Y$, but a union of subsets of this form cannot necessarily be written as a single subset of this form.

(j) Let $\{X_i\}_{i \in I}$ and $\{Y_i\}_{i \in I}$ be families of nonempty topological spaces, and $f_i : X_i \to Y_i$ a function for each i. Suppose that $\prod_{i \in I} X_i$ and $\prod_{i \in I} Y_i$ are both given the product topology, or both given the box topology. Then the function

$$\begin{aligned} \prod f_i : \prod_{i \in I} X_i &\longmapsto \prod_{i \in I} Y_i \\ (x_i)_{i \in I} &\longmapsto \big(f_i(x_i)\big)_{i \in I} \end{aligned}$$

is continuous if and only if f_i is continuous for each i.

True. *Hint:* See Homework 7 Warm-up Problem 3. Note that the product of functions $\prod f_i$ is not the same construction as a function $f : X \to \prod_{i \in I} Y_i$ with coordinate functions f_i.

(k) Let $C \subseteq \mathbb{R}^\omega$ be the set of all sequences $(a_n)_{n \in \mathbb{N}} \in \mathbb{R}^\omega$ that are eventually zero. Then C is closed in the uniform topology.

False. *Hint:* Sequences that converge to zero are in $\overline{C}$.

(l) If a subspace A of a space X is connected, then ∂A is connected.

False. *Hint:* Consider $(0, 1)$ with the standard topology.

(m) For any index set J, the product $(0, 1)^J$ in the product topology is connected.

True. *Hint:* An arbitrary product of path-connected spaces is path-connected, and hence connected.

(n) Let $p : X \to Y$ be a quotient map. If Y is a T_1-space, then so is X.

False. *Hint:* Consider any quotient map $\mathbb{R} \to \{0, 1\}$.

(o) Consider $X = \mathbb{R}$ with the discrete topology. Then its one-point compactification $\hat{X} = X \cup \{\infty\}$ has the topology $\{U \subseteq \hat{X} \mid \infty \notin U\} \cup \{\hat{X}\}$.

False. *Hint:* The topology on $\hat{X}$ is $\{U \subseteq \hat{X} \mid \infty \notin U \text{ or } U \text{ is cofinite}\}$. The proposed topology is compact but not Hausdorff.

(p) Let X be a locally compact Hausdorff space, and let $\hat{X}$ be its one-point compactification. Let f be a continuous map from X to a Hausdorff space Y. Then there is at most one value $\hat{f}(\infty) \in Y$ that would extend the function f to a continuous function $\hat{f} : \hat{X} \to Y$.

True. *Hint:* X is dense in $\hat{X}$. See Homework 5 Problem 4(c).

(q) The cofinite topology on $\mathbb{R}$ is separable but not second countable.

True. *Hint:* Any countably infinite subset is dense, but we proved in class that $(\mathbb{R}, \text{cofinite})$ is not second countable.

(r) The space $\mathbb{N}$ with the cofinite topology is a Baire space.

False. *Hint:* Each point $\{n\}$ is a closed subset with empty interior, but their countable union $\bigcup_{n\in\mathbb{N}}\{n\} = \mathbb{N}$ has nonempty interior in $\mathbb{N}$.

(s) Let X be a space. Then $\mathscr{C}(X, \mathbb{R})$ in the uniform topology is a Baire space.

True. *Hint:* Since $\mathbb{R}$ is complete, we proved that $\mathscr{C}(X, \mathbb{R})$ is complete with respect to the uniform metric. The Baire category theorem states that any complete metric space is a Baire space.

(t) The set of functions $\{f_n(x) = nx \mid n \in \mathbb{N}\}$ in $\mathscr{C}([0, 1], \mathbb{R})$ is equicontinuous.

False. *Hint:* Apply the definition of equicontinuous. Note that the sets $f_n^{-1}(B_\epsilon(0)) = B_{\frac{\epsilon}{n}}(0)$ have diameters tending to zero as n tends to infinity.

2. (20 points) Each of the following statements is either true or false. If the statement holds in general, write "`True`". Otherwise, state a counterexample. **No justification necessary.**

(i) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.

True. *Hint:* Consider the definition of closure.

(ii) If $A \subseteq B$, then $\partial A \subseteq \partial B$.

False. Take $A = \mathbb{Q}$ as a subspace of $B = \mathbb{R}$ in the standard topology. Then $\partial \mathbb{Q} = \mathbb{R}$ but $\partial \mathbb{R} = \varnothing$.

(iii) Let $f : X \to Y$ be a map of topological spaces. If f is continuous, then $\overline{f(A)} \subseteq f(\overline{A})$ for every $A \subseteq X$.

False. For f continuous, $f(\overline{A}) \subseteq \overline{f(A)}$, but equality does not hold in general. For example, consider constant map $f(x) = 0$ from ($\mathbb{R}$, indiscrete) to ($\mathbb{R}$, indiscrete). Then f is continuous, and $f(\overline{\mathbb{R}}) = f(\mathbb{R}) = \{0\}$, but $\overline{f(\mathbb{R})} = \overline{\{0\}} = \mathbb{R}$.

(iv) Let S be a subset of a topological space X, and suppose that the subspace topology on S is the discrete topology. Then S is closed.

False. Consider for example $S = \{\frac{1}{n} \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$. Then S is discrete but has $0 \in \overline{S}$, $0 \notin S$.

(v) Let $\{X_i\}_{i \in I}$ be a collection of metric spaces with the metric topology. Then their product $\prod_{i \in I} X_i$ is metrizable in the box topology.

False. We proved in class that the box topology on $\mathbb{R}^\omega$ is not metrizable.

(vi) Let A be a path-connected subspace of a space X. Then $\overline{A}$ is path-connected.

False. We proved that the graph of the function $\sin\left(\frac{1}{x}\right)$ on $(0, 1)$ is path-connected, but that its closure in $\mathbb{R}^2$ is not.

(vii) If X is a connected topological space, then so are all its quotient spaces.

True. *Hint:* The continuous image of a connected space is connected.

(viii) Let $f : X \to Y$ be an open map of topological spaces. Then the restriction of f to any subspace of X is open (with respect to the subspace topology).

False. This is true of open subspaces, but not in general. For example, the identity map $f : \mathbb{R} \to \mathbb{R}$ is open, but if we restrict it to the subspace $[0,1]$, then $[0,1]$ is open in the subspace topology on $[0,1]$, but its image $[0,1] \subseteq \mathbb{R}$ is not.

(ix) Let $f : X \to Y$ be a continuous invertible map, and suppose that on every connected component of X the map f restricts to a homeomorphism to a connected component of Y. Then f is a homeomorphism.

False. For example, the identity map ($\mathbb{Q}$, discrete) to ($\mathbb{Q}$, standard) is such a map – the connected components of both spaces are singleton points $\{q\}$, and vacuously the map restricts to a homeomorphism on single points – but this map is not a homeomorphism.

(x) There does not exist a path from a to d in the space $X = \{a, b, c, d\}$ with the topology $\Big\{\varnothing, \{a\}, \{d\}, \{a, d\}, \{c, d\}, \{a, c, d\}, X\Big\}$.

False. Consider the map $\gamma : [0,1] \to X$ defined by $\gamma(x) = \begin{cases} a, & x \in [0, \frac{1}{2}) \\ b, & x = \frac{1}{2} \\ d, & x \in (\frac{1}{2}, 1]. \end{cases}$

(xi) Any compact space X is second countable.

False. For example, ($\mathbb{R}$, cofinite) is compact but not second countable.

(xii) Any second countable space X is compact.

False. For example, ($\mathbb{R}$, standard) is second countable but not compact.

(xiii) If X is complete with respect to a metric d, then X is complete with respect to any metric equivalent to d.

False. For example, $\mathbb{R}$ is complete with the Euclidean metric. But $\mathbb{R}$ is homeomorphic to $(0,1)$, and so this homeomorphism defines an equivalent metric, which is not complete, on $\mathbb{R}$.

(xiv) If a space X is compact, then X is limit point compact.

True. Proved in class.

(xv) The continuous image of a regular space is regular.

False. For example, the standard topology on $\mathbb{R}$ is regular (since it is metrizable). The identity map $(\mathbb{R}, \text{standard}) \to (\mathbb{R}, \text{cofinite})$ is continuous since the standard topology is finer. But the cofinite topology on $\mathbb{R}$ is not regular; it is not even Hausdorff.

(xvi) Every discrete space is normal.

True. *Hint:* Discrete spaces are T_1. Every closed subset of a discrete space is also open.

(xvii) A subspace of a separable space is separable.

False. We saw in class that the Sorgenfrey plane $\mathbb{R}_\ell^2$ has countable dense subset $\mathbb{Q}^2$, but its antidiagonal $L = \{(-x, x) \mid x \in \mathbb{R}\}$ is an uncountable discrete subspace and therefore not separable.

(xviii) For a topological space X and a metric space Y the set of all continuous, bounded functions in Y^X is a closed subset of Y^X in the uniform topology.

True. Proved in class.

(xix) If $(f_n)_{n\in\mathbb{N}}$ is a convergent sequence of bounded continuous functions $\mathbb{R} \to \mathbb{R}$ in the compact-open topology, then the limit f is bounded.

False. Consider the sequence of functions $f_n(x) = \begin{cases} x, & x \in [-n, n] \\ -n, & x < -n \\ n, & x > n. \end{cases}$

This is a sequence of continuous bounded functions that converges uniformly on compact sets to the unbounded function $f(x) = x$.

(xx) Consider the set $\mathscr{F} = \{f : [0,1] \to \mathbb{R} \mid f(x) = ax^2 + bx + c \text{ with } a, b, c \in [0,1]\}$ of continuous functions. Then any sequence of functions in $\mathscr{F}$ has a subsequence that converges uniformly to some continuous function $[0,1] \to \mathbb{R}$.

True. *Hint:* The set $\mathscr{F}$ is pointwise bounded and equicontinuous, so by Ascoli's theorem (Homework 14 Problem 2(g)) it has (sequentially) compact closure.

3. (a) (2 points) For each of the following sequences of elements of $\mathbb{R}^\omega$, circle all the topologies on $\mathbb{R}^\omega$ with respect to which the sequence converges. **No justification needed.**

(i)

$$\begin{aligned} \mathbf{a_1} &= (1,1,1,1,1,\dots) \\ \mathbf{a_2} &= (0,2,2,2,2,\dots) \\ \mathbf{a_3} &= (0,0,3,3,3,\dots) \\ \mathbf{a_4} &= (0,0,0,4,4,\dots) \\ &\vdots \end{aligned}$$

`product topology`

`uniform topology`

`box topology`

(ii)

$$\begin{aligned} \mathbf{b_1} &= (1,1,1,1,1,\dots) \\ \mathbf{b_2} &= (0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\dots) \\ \mathbf{b_3} &= (0,0,\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},\dots) \\ \mathbf{b_4} &= (0,0,0,\tfrac{1}{4},\tfrac{1}{4},\dots) \\ &\vdots \end{aligned}$$

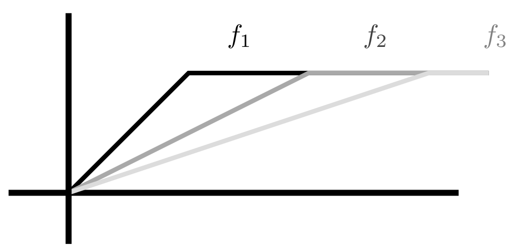
`product topology`

`uniform topology`

`box topology`

(b) (2 points) For each of the sequences of continuous functions $f_n : [0,\infty) \to \mathbb{R}$, circle all the topologies on $\mathscr{C}([0,\infty),\mathbb{R})$ with respect to which the sequence converges.

(i) $f_n(x) = \begin{cases} \frac{x}{n}, & x \in [0,n] \\ 1, & x > n \end{cases}$

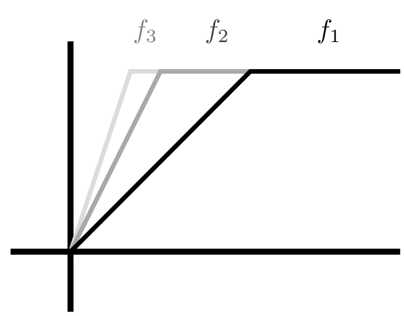


`topology of pointwise convergence`

`compact-open topology`

`uniform topology`

(ii) $f_n(x) = \begin{cases} nx, & x \in [0,\frac{1}{n}] \\ 1, & x > \frac{1}{n} \end{cases}$



`topology of pointwise convergence`

`compact-open topology`

`uniform topology`

(c) (2 points) Circle all terms that apply. The topology $\{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\mathbb{R}\} \cup \{\varnothing\}$ on $\mathbb{R}$ is ...

`compact` `path-connected` `separable` T_4 `(normal)`

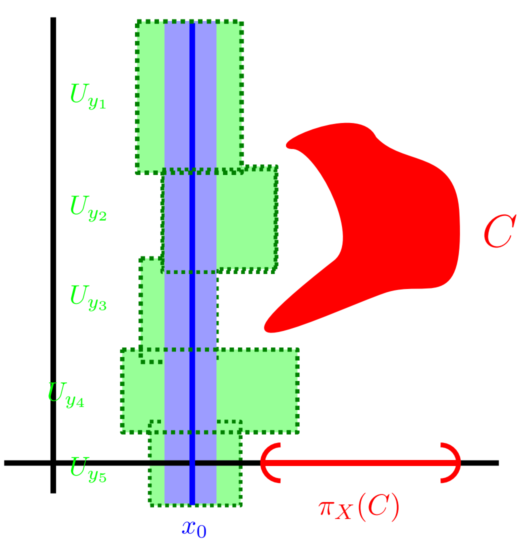
4. (4 points) Let X and Y be topological spaces. We saw in class that, in general, the projection map $\pi_X : X \times Y \to X$ may not be closed. Show that, if Y is compact, then π_X is a closed map.

Solution. Suppose that Y is compact, and let $C \subseteq X \times Y$ be any closed subset. To prove that $\pi_X(C)$ is closed in X, we will show that its complement is open. So let $x_0 \in X$ be any point in the complement; we wish to show that x_0 is an interior point of the complement.

Since $x_0 \notin \pi_X(C)$, the preimage $\pi^{-1}(\{x_0\}) = \{x_0\} \times Y$ must be contained in the complement of C. Hence, for each point $y \in Y$, there is some open neighbourhood of (x_0, y) of the form $U_y \times V_y$ (for $U_y \subseteq X$ and $V_y \subseteq Y$ open) that is contained in $(X \times Y) \setminus C$.

Since $y \in V_y$ for any $y \in Y$, the sets $\{V_y\}_{y \in Y}$ must cover Y. Because Y is compact, then, there is some finite subcover $V_{y_1}, \ldots, V_{y_n}$ that covers Y.

Then consider the set $U = \pi_X((U_{y_1} \cap \cdots \cap U_{y_n}) \times Y)$. Since π_X is an open map, this set is open, and it contains the point x_0 by construction. So it suffices to show that it is disjoint from $\pi_X(C)$.



Let $x \in \pi_X((U_{y_1} \cap \cdots \cap U_{y_n})) \times Y)$. To show that $x \notin \pi_X(C)$, we must show that $(x, y) \notin C$ for all y.

Fix y. By choice of x,

$$(x, y) \in \Big((U_{y_1} \cap \cdots \cap U_{y_n}) \times Y\Big).$$

Then $y \in V_{y_i}$ for some i (since these sets cover Y), and $x \in (U_{y_1} \cap \cdots \cap U_{y_n}) \subseteq U_{y_i}$. So $(x, y) \in U_{y_i} \times V_{y_i}$. But the subset $U_{y_i} \times V_{y_i}$ is contained in the complement of C by assumption. We conclude that (x, y) is not in C. Since this is true for all y, we see that x is not in $\pi_X(C)$. Hence $U \cap \pi_X(C) = \varnothing$.

It follows that x_0 is an interior point of the complement of $\pi_X(C)$, and therefore that $\pi_X(C)$ is closed. Thus, π_X is a closed map.

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