Midterm Exam

Math 590
13 March 2019
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Name: ______________________________

Instructions: This exam has 4 questions for a total of 20 points.

The exam is closed-book. No books, notes, cell phones, calculators, or other devices are permitted. Scratch paper is available.

Fully justify your answers unless otherwise instructed. You may quote any results proved in class or on the homeworks without proof, but please include a precise statement of the result you are quoting.

You have 50 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

Question	Points	Score
1	7	
2	8	
3	2	
4	3	
Total:	20	

1. (7 points) Each of the following statements is either true or false. If the statement holds in general, circle "**T**". Otherwise, circle "**F**". **No justification necessary.**

   (a) For sets A, B, C, D, there is equality $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$. **T** (**F**)

   Hint: Consider both sides of the equation in the case that A, B, C, D are each a single point.

   (b) The set of all functions $\{f : \mathbb{N} \to \mathbb{N}\}$ is countable. **T** (**F**)

   Hint: A function $f : \mathbb{N} \to \mathbb{N}$ can be viewed as a sequence of natural numbers. Use Cantor's diagonalization argument.

   (c) Consider $\mathbb{R}$ with the cofinite topology. Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = \sin(x)$. Then f is continuous. **T** (**F**)

   Hint: The set $\{0\} \subseteq \mathbb{R}$ is closed. What is its preimage?

   (d) Let X be a metric space and $B_r(x)$ a ball of radius r in X. Then any two points in $B_r(x)$ are distance less than $2r$ apart. (**T**) **F**

   Hint: Triangle inequality.

   (e) Let X and $X_i, (i \in I)$, be topological spaces. A function $f : X \to \prod_{i\in I} X_i$ is continuous with respect to the product topology on $\prod_{i\in I} X_i$ if and only if all of its coordinate functions are continuous. (**T**) **F**

   Hint: Homework #6 Problem 5(c).

   (f) Let X be a metric space, and let Y be a bounded metric space. If X is homeomorphic to Y (with respect to the metric topologies), then X is also bounded. **T** (**F**)

   Hint: Consider the sets $X = \mathbb{R}$ and $Y = (0, 1)$ with the Euclidean metric.

   (g) The set $\mathbb{R}$ with the cofinite topology is connected. (**T**) **F**

   Hint: If A and B separate $\mathbb{R}$, then A and B are nonempty disjoint open sets.

2. (8 points) Each of the following statements is either true or false. If the statement holds in general, write "`True`". Otherwise, state a counterexample. **No justification necessary.** You can get partial credit for correctly writing "`False`" without a counterexample.

   (a) If C is a closed set, then C is the closure of some open set U.

   False. Consider the set $\{0\}$ in $\mathbb{R}$ (with the standard topology). Then $\{0\}$ is closed, but its only open subset is $\varnothing$, whose closure is $\varnothing$.

   (b) Suppose that $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence in a topological space X, and $f : X \to Y$ is a continuous function. Then $\Big(f(a_n)\Big)_{n\in\mathbb{N}}$ is a convergent sequence in Y.

   True. *Hint:* Homework #5 Problem 2(b).

   (c) Let $\{X_i\}_{i\in I}$ be a collection of T_1-spaces. Then the product topology on $\prod_{i\in I} X_i$ has the T_1 property.

   True. *Hint:* It suffices to show that points are closed. See Homework #7 Problem 1(c).

   (d) Let X be a topological space and A a subspace. If C is closed in A, then C is closed in X.

   False. Consider the set $A = (0, 2)$ in $\mathbb{R}$ (with the standard topology). Then $C = (0, 1] = A \cap [-1, 1]$ is closed in A, but it is not closed in $\mathbb{R}$.

   (e) Let $f : \mathbb{R} \to Y$ be a continuous function from $\mathbb{R}$ (with the standard topology) to a Hausdorff space Y. Then f is completely determined by its values on $\mathbb{Q} \subseteq \mathbb{R}$.

   True. *Hint:* Homework #5 Problem 4(c).

   (f) Let X be a topological space with the property that a sequence $(a_n)_{n\in\mathbb{N}}$ converges in X if and only if the sequence is eventually constant. (Recall that this means that there is some $N \geq 0$ so that $a_n = a_N$ for all $n \geq N$.) Then X has the discrete topology.

   False. This condition holds for $X = \mathbb{R}$ with the cocountable topology. For any $c \in \mathbb{R}$, the set $\mathbb{R} \setminus \{a_n \mid n \in \mathbb{N}, a_n \neq c\}$ is a neighbourhood of c. The sequence can only converge to c if this neighbourhood contains a_n for all but finitely many n, which requires $a_n = c$ for all but finitely many values of n.

   (g) Let $p : X \to A$ be a quotient map. If X is metrizable, then A is metrizable.

   False. Consider $X = \mathbb{R}$ with standard topology, which is induced by the Euclidean metric. Let $p : \mathbb{R} \to \{0, 1\}$ be the map $p(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1, & x \in [0, \infty). \end{cases}$ Then the quotient topology on $\{0, 1\}$ is $\{\varnothing, \{0\}, \{0, 1\}\}$, which is cannot be metrizable because it is not Hausdorff.

   (h) Let A and B be nonempty subsets of a topological space. If $A \cap B = \varnothing$, then $A \cup B$ is disconnected.

   False. For example, consider $A = (-\infty, 0)$ and $B = [0, \infty)$ as subsets of $\mathbb{R}$ with the Euclidean metric. Then A and B are disjoint but $A \cup B = \mathbb{R}$ is connected.

3. (2 points) Consider the standard topology on $\mathbb{R}$. Recall that $\mathbb{R}^\omega$ denotes the space $\prod_{\mathbb{N}} \mathbb{R}$ of sequences of real numbers. Let $X \subseteq \mathbb{R}^\omega$ be the set of all sequences that are eventually zero. (Recall that $(a_n)_{n\in\mathbb{N}}$ is *eventually zero* if there is some $N \geq 0$ so that $a_n = 0$ for all $n \geq N$.) State the following. **No justifiication needed.**

Closure of X in product topology: $\mathbb{R}^\omega$

Closure of X in box topology: X

4. (3 points) Let $p : X \to Y$ be a continuous map. Suppose that there exists a continuous map $f : Y \to X$ such that $p \circ f$ is the identity function on Y. Prove that p is a quotient map.

Solution: To verify that p is a quotient map, we must check three things,

- p is sujective,
- p is continuous (which is an assumption),
- if $p^{-1}(U)$ is open for some $U \subseteq Y$, then U is open.

First we check that p is surjective. Let $y \in Y$; we will show that y is in the image of p. Since $p \circ f$ is the identity on Y, we know that

$$(p \circ f)(y) = p(f(y)) = y.$$

Thus p maps $f(y) \in X$ to y in Y, so $y \in p(X)$ and we conclude that p is surjective.

We next check that $U \subseteq Y$ is open whenever $p^{-1}(U)$ is open. So suppose that $p^{-1}(U)$ is open for some $U \subseteq Y$. Because the function f is continuous, it follows that $f^{-1}\Big(p^{-1}(U)\Big)$ is also open. But

$$\begin{aligned} f^{-1}\Big(p^{-1}(U)\Big) &= \{y \in Y \mid f(y) \in p^{-1}(U)\} \\ &= \{y \in Y \mid p\Big(f(y)\Big) \in U\} \\ &= \{y \in Y \mid (p \circ f)(y) \in U\} \\ &= (p \circ f)^{-1}(U) \end{aligned}$$

But $p \circ f$ is the identity function on Y, so $(p \circ f)^{-1}(U) = U$. So we conclude that U is open, as desired. This concludes the proof that p is a quotient map.