Name: ______________________________ Score (Out of 9 points):

1. (6 points) State which of the following sets is countable by circling either "Countable" or "Uncountable". No justifcation necessary.

• The set $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$	(Countable)	Uncountable
• The set of all finite subsets of $\mathbb{Q}$	(Countable)	Uncountable
• The set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$	Countable	(Uncountable)
• The set of all sequences $(a_n)_{n\in\mathbb{N}}$ with $a_n \in \mathbb{Z}$ for all n	Countable	(Uncountable)
• The set of monic polynomials with integer coefficients $\{x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \mid a_i \in \mathbb{Z}, n \in \mathbb{N}\}$	(Countable)	Uncountable
• The set of all real numbers x that can be represented by a decimal expansion involving only the digits 0 and 1	Countable	(Uncountable)

2. (3 points) Let $S = \{0, 1\}$ and let S^ω denote the Cartesian product

$$S^\omega = S \times S \times S \times S \times \cdots$$

of a countably infinite number of copies of S. (So an element of S^ω is an ordered tuple such as

$$(0, 0, 1, 0, 1, 1, 1, 0, 1, 0, 1, 1, 0, 0, 0, \cdots)\Big).$$

Show that S^ω is not countable.

To show that this set is not countable, we will assume (for contradiction) that there exists a surjection $f : \mathbb{N} \to S^\omega$, but then exhibit an element of S^ω not in the image of f. Let

$$\begin{aligned} f(1) &= (a_{1,1}, a_{1,2}, a_{1,3}, a_{1,4}, \dots, a_{1,n}, \dots) \\ f(2) &= (a_{2,1}, a_{2,2}, a_{2,3}, a_{2,4}, \dots, a_{2,n}, \dots) \\ f(3) &= (a_{3,1}, a_{3,2}, a_{3,3}, a_{3,4}, \dots, a_{3,n}, \dots) \\ &\vdots \\ f(m) &= (a_{m,1}, a_{m,2}, a_{m,3}, a_{m,4}, \dots, a_{m,n}, \dots) \\ &\vdots \end{aligned}$$

where $a_{i,j} \in \{0, 1\}$ for all i, j. Then consider the element $B = (b_1, b_2, b_3, \dots) \in S^\omega$ constructed so that $b_i = \begin{cases} 0, & a_{i,i} = 1 \\ 1, & a_{i,i} = 0. \end{cases}$ Then B cannot equal $f(m)$ for any $m \in \mathbb{N}$, since B and $f(m)$ differ in the m^{th} coordinate $b_m \neq a_{m,m}$. Hence f does not surject, and S^ω cannot be countable.