Name: ____________________ Score (Out of 7 points):

1. (3 points) Let X be a set and let $p \in X$. Prove that the following is a topology on X:

$$\mathcal{T} = \{X\} \cup \{U \subseteq X \mid p \notin U\}.$$

Solution: We need to check the three conditions.

- $X \in \mathcal{T}$ by construction, and $\varnothing \in \mathcal{T}$ since $p \notin \varnothing$.

- Let $\{U_\alpha\}_{\alpha \in I} \subseteq \mathcal{T}$. If $U_\alpha = X$ for some index α, then $\bigcup_{\alpha \in I} U_\alpha = X \in \mathcal{T}$.
  If U_α is not X for any index α, then $p \notin U_\alpha$ for any $\alpha \in I$. It follows that $p \notin \bigcup_{\alpha \in I} U_\alpha$, so $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$.

- Suppose that $U, V \in \mathcal{T}$. If $U = V$ then $U \cap V = U \in \mathcal{T}$. If $U \neq V$, then at least one of U and V does not contain p, so $p \notin U \cap V$, and $U \cap V \in \mathcal{T}$.

We conclude that $\mathcal{T}$ satisfies the definition of a topology on X.

2. (4 points) Let $n \in \mathbb{N}$ and let $(X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2), \dots, (X_n, \mathcal{T}_n)$ be topological spaces. Consider the topology $\mathcal{T}$ on $X_1 \times X_2 \times \cdots \times X_n$ generated by the basis

$$\mathcal{B} = \{U_1 \times U_2 \times \cdots \times U_n \mid U_i \in \mathcal{T}_i \text{ for all } i\}.$$

(You do not need to show that this is a basis.) Let $(X, \mathcal{T}_X)$ be a topological space, and for each $i = 1, \dots, n$ let $f_i : X \to X_i$ be a function. Define

$$\begin{aligned} f : X &\longrightarrow X_1 \times X_2 \times \cdots \times X_n \\ x &\longmapsto (f_1(x), f_2(x), \dots, f_n(x)) \end{aligned}$$

Show that the function f is continuous (with respect to the topologies $\mathcal{T}_X$ and $\mathcal{T}$) if and only if each coordinate function f_i is continuous (with respect to the topologies $\mathcal{T}_X$ and $\mathcal{T}_i$).

Solution: Let us first suppose that the functions $f_i : X \to X_i$ are continuous for each i. This means that, for every open subset $U \subseteq X_i$, the subset $f_i^{-1}(U)$ is open. To check that f is continuous, by Homework #3 Problem 6, it suffices to check that $f^{-1}(B)$ is open for every basis element $B \subseteq X_1 \times X_2 \times \cdots \times X_n$. So choose an arbitrary basis element $B = U_1 \times U_2 \times \cdots \times U_n$. Then

$$\begin{aligned} f^{-1}(U_1 \times U_2 \times \cdots \times U_n) &= \{x \in X \mid f(x) \in U_1 \times U_2 \times \cdots \times U_n\} \\ &= \{x \in X \mid f_1(x) \in U_1, \text{ and } f_2(x) \in U_2, \dots, \text{ and } f_n(x) \in U_n\} \\ &= \{x \in X \mid x \in f_1^{-1}(U_1), \text{ and } x \in f_2^{-1}(U_2), \dots, \text{ and } x \in f_n^{-1}(U_n)\} \\ &= f_1^{-1}(U_1) \cap f_2^{-1}(U_2) \cap \cdots \cap f_n^{-1}(U_n). \end{aligned}$$

By assumption, this is a finite intersection of open sets, and therefore is open. We conclude that f is continuous.

Next assume that f is continuous, and fix $i \in \{1, 2, \dots, n\}$. Let U be an open set in X_i. Since f is continuous, the preimage under f of the open set

$$X_1 \times X_2 \times \cdots \times X_{i-1} \times U \times X_{i+1} \times \cdots \times X_n$$

is open in X. But this preimage is

$$\begin{aligned} &f^{-1}(X_1 \times X_2 \times \cdots \times X_{i-1} \times U \times X_{i+1} \times \cdots \times X_n) \\ &= f_1^{-1}(X_1) \cap f_2^{-1}(X_2) \cap \cdots \cap f_{i-1}^{-1}(X_{i-1}) \times f_i^{-1}(U) \cap f_{i+1}^{-1}(X_{i+1}) \cap \cdots \cap f_n^{-1}(X_n) \\ &= X \cap X \cap \cdots \cap X \cap f_i^{-1}(U) \cap X \cap \cdots \cap X \\ &= f_i^{-1}(U). \end{aligned}$$

Thus $f_i^{-1}(U)$ is open, and we conclude that f_i is continuous.

Alternate argument: Next assume that f is continuous, and fix $i \in \{1, 2, \dots, n\}$. Then f_i is the composition $\pi_i \circ f$ of the continuous function f and the continuous projection map $\pi_i : X_1 \times X_2 \times \cdots \times X_n \to X_i$, and is therefore continuous.