Name: ______________________________ Score (Out of 8 points):

1. (4 points) Find the set of all limits of the following sequences. If the sequences does not converge to any point, write `"Does not converge"`. **No justification necessary.**

- Let $X = \{a, b, c, d\}$ have the topology $\mathcal{T} = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c, d\}\}$.

  (i) $a,\ b,\ a,\ b,\ a,\ b,\ a,\ b,\ \cdots$ **Solution:** $\{c, d\}$

- Let $\mathbb{R}$ have the topology $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\varnothing\} \cup \{\mathbb{R}\}$.

  (ii) $0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ \cdots$ **Solution:** $(-\infty, 0]$

  (iii) $(-n)_{n\in\mathbb{N}}$ **Solution:** Does not converge

- Let $\mathbb{R}$ have the topology $\mathcal{T} = \{\varnothing\} \cup \{U \subseteq \mathbb{R} \mid 0 \in U\}$.

  (iv) $0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ \cdots$ **Solution:** $\mathbb{R}$

2. (4 points) Show that a topological space X is Hausdorff if and only if, for each $x \in X$,

$$\bigcap_{U \text{ a neighbourhood of } x} \overline{U} = \{x\}.$$

Solution. Since $x \in U \subseteq \overline{U}$ for all neighbourhoods U of x, it is clear that

$$\bigcap_{U \text{ a neighbourhood of } x} \overline{U} \supseteq \{x\}$$

for any topological space X. Hence, our goal is to show that

$$\bigcap_{U \text{ a neighbourhood of } x} \overline{U} \subseteq \{x\}$$

if and only if X is Hausdorff.

First suppose that X is Hausdorff. This means, for any $x, y \in X$ with $y \neq x$, there are disjoint neighbourhoods U_x of x and U_y of y. But, since y has a neighbourhood that does not intersect U_x, we can conclude that $y \notin \overline{U_x}$. Hence for all $y \neq x$,

$$y \notin \bigcap_{U \text{ a neighbourhood of } x} \overline{U}.$$

We conclude that, if X is Hausdorff, then for any $x \in X$,

$$\bigcap_{U \text{ a neighbourhood of } x} \overline{U} \subseteq \{x\}.$$

Next suppose that

$$\bigcap_{U \text{ a neighbourhood of } x} \overline{U} \subseteq \{x\}$$

for all $x \in X$, and consider any pair of distinct points $x, y \in X$. By assumption, there must be some neighbourhood U_x of x so that $y \notin \overline{U_x}$. This means that there is some neighbourhood U_y of y that does not intersect U_x. Thus x and y have disjoint neighbourhoods U_x and U_y, and we conclude that X is Hausdorff as desired.