Name: ______________________ Score (Out of 5 points):

1. (1 point) Let $p : \mathbb{R} \to \{a, b, c, d\}$ be the following map from the $\mathbb{R}$ (with the standard topology) to the set $\{a, b, c, d\}$,

$$p : \mathbb{R} \longrightarrow \{a, b, c, d\}$$
$$p(x) = \begin{cases} a, \ x \in (-\infty, 1) \\ b, \ x = 1, 2 \\ c, \ x \in (1, 2) \cup (2, 3) \\ d, \ x \in [3, \infty). \end{cases}$$

Write the induced quotient topology on $\{a, b, c, d\}$.

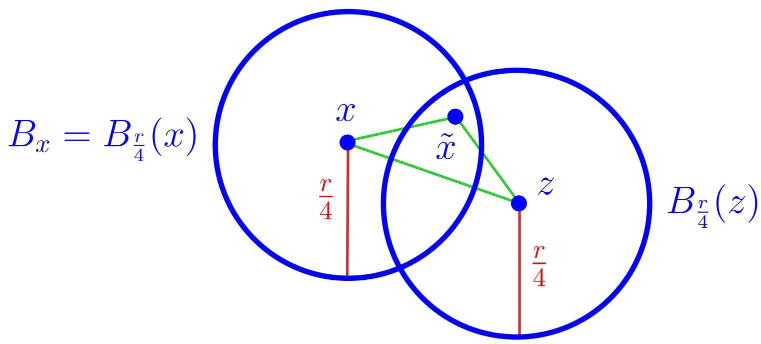
Solution. The quotient topology is $\Big\{\varnothing, \{a\}, \{c\}, \{a, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{a, b, c, d\}\Big\}$.

2. (4 points) Let (X, d) be a metric space with at least two elements. Show that there exist nonempty open sets in X whose closures are disjoint.

Solution. Suppose that X contains the two distinct elements x and y, and suppose that $d(x, y) = r$. Then $r > 0$ by definition of a metric, and so the sets $B_x = B_{\frac{r}{4}}(x)$ and $B_y = B_{\frac{r}{4}}(y)$ are open balls around x and y, respectively. We will show that these two nonempty open sets have disjoint closure.

Suppose (for the sake of contradiction) that z were an element in $\overline{B_x}$ and $\overline{B_y}$. This means that every open neighbourhood U_z of z contains a point in B_x and contains a point in B_y. So consider the open neighbourhood $U_z = B_{\frac{r}{4}}(z)$.

By assumption this neighbourhood contains a point $\tilde{x} \in B_x$.



But then observe that

$$\begin{aligned} d(x, z) &\leq d(x, \tilde{x}) + d(\tilde{x}, z) \\ &< \frac{r}{4} + \frac{r}{4} \qquad \text{(since } \tilde{x} \in B_{\frac{r}{4}}(x) \text{ and } \tilde{x} \in B_{\frac{r}{4}}(z)) \\ &= \frac{r}{2} \end{aligned}$$

Since $B_{\frac{r}{4}}(z)$ must also contain a point of B_y, the same argument shows that $d(y, z) < \frac{r}{2}$.

But then

$$\begin{aligned} d(x, y) &\leq d(x, z) + d(z, y) \\ &< \frac{r}{2} + \frac{r}{2} \\ &= r \end{aligned}$$

which contradicts our premise that $d(x, y) = r$. Thus no such element z can exist, and we conclude that $\overline{B_x} \cap \overline{B_y} = \varnothing$ as claimed.