Name: ______________________________ Score (Out of 7 points):

1. (3 points) Let $X = \{a, b, c\}$. For each of the following topologies on X, write down a path from a to c, if one exists. Otherwise, write `"no path exists"`.

   (a) $\mathcal{T} = \Big\{\varnothing, \{a\}, \{b, c\}, \{a, b, c\}\Big\}$

   Solution: No such path exists.

   Hint: The preimage of $\{a\}$ would be a proper nonempty subset of the domain that is both open and closed.

   (b) $\mathcal{T} = \Big\{\varnothing, \{a\}, \{a, b\}, \{a, b, c\}\Big\}$

   Solution: Consider the path

$$\gamma : [0,1] \to \mathbb{R}$$
$$\gamma(t) = \begin{cases} a, \ t \in \left[0, \frac{1}{2}\right), \\ c, \ t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

   (c) $\mathcal{T} = \Big\{\varnothing, \{a\}, \{c\}, \{a, c\}, \{a, b, c\}\Big\}$

   Solution: Consider the path

$$\gamma : [0,1] \to \mathbb{R}$$
$$\gamma(t) = \begin{cases} a, \ t \in \left[0, \frac{1}{2}\right), \\ b, \ t = \frac{1}{2}, \\ c, \ t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

2. (4 points) Prove the following result.

   Theorem. A space X is locally path-connected if and only if for every open set U of X, each path component of U is open in X.

Solution: First, suppose that X is locally path-connected.

Let U be a subset of X, and let C be a path component of U. We wish to show that C is open. So let $x \in C$; we will show that x is an interior point of C.

Since U is a neighbourhood of x, by definition of local path-connectedness there exists some open, path-connected set V such that $x \in V \subseteq U$. But since V is a path-connected subset of U, V must be contained in the path component C of x in U. Thus $x \in V \subseteq C$, and we conclude that x is an interior point of C. Therefore C is open, as claimed.

Next suppose that, for every open set U of X, each path-component of U is open in X. We will show that X is locally path-connected.

So let $x \in X$, and let U be a neighbourhood of x. Let C be the path-component of U containing x. Then C is open in U by assumption, and (because U is open) it follows that C is open in X. Moreover, C is path-connected by definition of path-component. Hence $V = C$ is a path-connected neighbourhood of x contained in U. We conclude that X is locally path-connected, as claimed.