Name: ____________________ Score (Out of 12 points):

1. (8 points) State which of the following topologies on $\mathbb{R}$ are "`first countable`" , "`second countable`", or neither, by circling the appropriate term(s). No justifcation necessary.

- ($\mathbb{R}$, standard topology) — **(first countable)** **(second countable)**
- ($\mathbb{R}$, discrete topology) — **(first countable)** second countable
- ($\mathbb{R}$, indiscrete topology) — **(first countable)** **(second countable)**
- $(\mathbb{R}, \{(a,\infty) \mid a \in \mathbb{R}\} \cup \{\varnothing\} \cup \{\mathbb{R}\})$ — **(first countable)** **(second countable)**
- $(\mathbb{R}, \{U \mid 0 \notin U\} \cup \{\mathbb{R}\})$ — **(first countable)** second countable
- ($\mathbb{R}$, cofinite) — first countable second countable
- ($\mathbb{R}$, cocountable) — first countable second countable
- ($\mathbb{R}$, basis $\{[a,b) \mid a,b \in \mathbb{R}\}$) — **(first countable)** second countable

2. (4 points) Let X be a compact metric space. Prove that X is second countable.

Solution. Fix n, and consider the cover of X by the balls $\left\{B_{\frac{1}{n}}(x) \mid x \in X\right\}$. Since X is compact, there is a finite subcover by balls

$$B_{\frac{1}{n}}(x_{n,1}), B_{\frac{1}{n}}(x_{n,2}), \dots B_{\frac{1}{n}}(x_{n,N_n}).$$

We claim that the set

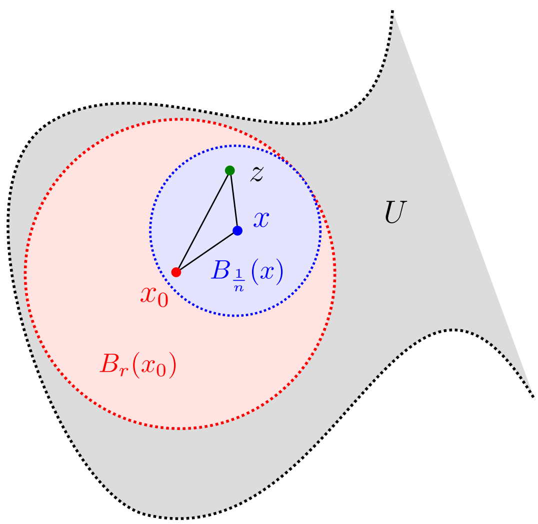
$$\mathcal{B} = \left\{B_{\frac{1}{n}}(x_{n,i}) \;\middle|\; n \in \mathbb{N},\ i = 1, \dots, N_n\right\}$$

is a countable basis for X. The set $\mathcal{B}$ is countable since it is indexed by a countable union of the finite sets $\{1, 2, \dots, N_n\}$. We must prove that it generates the metric topology.

To show that $\mathcal{B}$ is a basis for the metric topology, it suffices to show the following: for every open subset U of X and every point x_0 in U, there is some basis element $B \in \mathcal{B}$ such that $x_0 \in B \subseteq U$.

So let U be an open subset in X, and let $x_0 \in U$. Then we proved that there exists some $r > 0$ so that $B_r(x_0) \subseteq U$.

Choose n large enough that $\frac{1}{n} < \frac{r}{2}$, and choose an element of $\mathcal{B}$ of the form $B_{\frac{1}{n}}(x)$ that contains x_0. Such a basis element exists, since by construction the balls of radius $\frac{1}{n}$ cover X.



Then, by the triangle equality, any point z in $B_{\frac{1}{n}}(x)$ is distance at most r from x_0:

$$d(x_0, z) \;\leq\; d(x_0, x) + d(x, z) \;<\; \frac{1}{n} + \frac{1}{n} \;=\; \frac{2}{n} \;<\; r.$$

So $z \in B_r(x_0)$.

This implies that $x_0 \in B_{\frac{1}{n}}(x) \subseteq B_r(x_0) \subseteq U$, which concludes the proof that $\mathcal{B}$ generates the metric topology on X.