Name: ______________________________ Score (Out of 8 points):

1. Let (X, d) be a metric space.

   (a) (2 points) Let $\mathscr{B}(X, \mathbb{R})$ denote the set of bounded functions $f : X \to \mathbb{R}$, where $\mathbb{R}$ has the Euclidean metric. Explain how to combine our results from class to show that $\mathscr{B}(X, \mathbb{R})$ is a complete metric space with respect to the sup metric ρ.

   (b) (3 points) Fix $x_0 \in X$. For each $a \in X$, define a function

$$\phi_a : X \to \mathbb{R}$$
$$\phi_a(x) = d(x, a) - d(x, x_0).$$

   Use the triangle inequality to show that $|d(x, a) - d(x, x_0)| \leq d(a, x_0)$. Conclude that ϕ_a is bounded.

(c) (3 points) Show that the function

$$\begin{aligned} \Phi &: X \to \mathscr{B}(X, \mathbb{R}) \\ \Phi(a) &= [\phi_a : X \to \mathbb{R}] \end{aligned}$$

defines an isometric embedding of X into the complete metric space $\mathscr{B}(X, \mathbb{R})$ with the sup metric ρ. Recall that this means that, for all $a, b \in X$,

$$d(a, b) = \rho(\phi_a, \phi_b).$$