Name: ______________________________ Score (Out of **??** points):

1. Let (X, d) be a metric space.

   (a) (2 points) Let $\mathscr{B}(X, \mathbb{R})$ denote the set of bounded functions $f : X \to \mathbb{R}$, where $\mathbb{R}$ has the Euclidean metric. Explain how to combine our results from class to show that $\mathscr{B}(X, \mathbb{R})$ is a complete metric space with respect to the sup metric ρ.

   Solution: We proved in class that $\mathbb{R}$ is complete, and therefore that the space of functions $\mathbb{R}^X$ is complete with respect to the uniform metric $\bar{\rho}$. We proved that the subspace $\mathscr{B}(X, \mathbb{R})$ of bounded functions is closed in $\mathbb{R}^X$, and moreover that a closed subset of a complete metric space is also complete. Finally, we saw that on $\mathscr{B}(X, \mathbb{R})$,

   $$\bar{\rho}(f, g) = \min\{\rho(f, g), 1\},$$

   and that it follows that $\mathscr{B}(X, \mathbb{R})$ is complete with respect to the sup metric if and only if it is complete with respect to the uniform metric.

   (b) (3 points) Fix $x_0 \in X$. For each $a \in X$, define a function

   $$\begin{aligned} \phi_a &: X \to \mathbb{R} \\ \phi_a(x) &= d(x, a) - d(x, x_0). \end{aligned}$$

   Use the triangle inequality to show that $|d(x, a) - d(x, x_0)| \leq d(a, x_0)$. Conclude that ϕ_a is bounded.

   Solution: By the triangle inequality,

   $$\begin{aligned} d(x, a) &\leq d(x, x_0) - d(x_0, a) \qquad \text{so} \\ d(x_0, a) &\geq d(x, x_0) - d(x, a). \end{aligned}$$

   It also follows from the triangle inequality that

   $$\begin{aligned} d(x, x_0) &\leq d(x, a) - d(a, x_0) \qquad \text{so} \\ d(x_0, a) &\geq d(x, a) - d(x, x_0). \end{aligned}$$

   Combining these results implies $d(x_0, a) \geq |d(x, a) - d(x, x_0)|$, as claimed.

   Thus $|\phi_a(x)| \leq d(x_0, a)$ for all $x \in X$. This implies that the image $\phi_a(X)$ is contained in the closed interval $[-d(x_0, a), d(x_0, a)] \subseteq \mathbb{R}$ and hence is bounded.

(c) (3 points) Show that the function

$$\Phi : X \to \mathscr{B}(X, \mathbb{R})$$
$$\Phi(a) = [\phi_a : X \to \mathbb{R}]$$

defines an isometric embedding of X into the complete metric space $\mathscr{B}(X, \mathbb{R})$ with the sup metric ρ. Recall that this means that, for all $a, b \in X$,

$$d(a, b) = \rho(\phi_a, \phi_b).$$

Solution: We proved in part (b) that this function ϕ_a is bounded for each a, which shows that the map Φ is well-defined. It remains to show that it is an isometric embedding.

The sup metric ρ is defined by the equation

$$\begin{aligned}\rho(\phi_a, \phi_b) &= \sup_{x \in X} |\phi_a(x) - \phi_b(x)| \\ &= \sup_{x \in X} |d(x, a) - d(x, x_0) - d(x, b) + d(x, x_0)| \\ &= \sup_{x \in X} |d(x, a) - d(x, b)|\end{aligned}$$

But then replacing x_0 by b in the computation in part (b), we conclude that

$$|d(x, a) - d(x, b)| \leq d(a, b),$$

so

$$\begin{aligned}\rho(\phi_a, \phi_b) &= \sup_{x \in X} |d(x, a) - d(x, b)| \\ &\leq \sup_{x \in X} |d(a, b)| \\ &= d(a, b).\end{aligned}$$

To obtain a lower bound on the superemum, observe that when $x = a$,

$$|\phi_a(x) - \phi_b(x)| = |d(x, a) - d(x, b)| = |d(a, a) - d(a, b)| = d(a, b).$$

We conclude that $\rho(\phi_a, \phi_b) = d(a, b)$.