This worksheet contains a number of review problems to practice for the final. Students are **not** responsible for knowing any new definitons or results introduced on this handout. Correspondingly, you may **not** quote these results on the final without proof.

1. Let (X, d) be a metric space, and let A be a set. Let $f : A \to X$ be an injective function. Prove that the function f allows us to define a metric D on A, given by $D(a, b) = d(f(a), f(b))$.

2. Let X be a metric space, and $A \subseteq X$. Prove that $\overline{A} = \left\{ x \in X \;\middle|\; \inf_{a \in A} d(x, a) = 0 \right\}$.

3. Let X be a metric space.
   (a) Show that the union of a finite number of balls in X is bounded.
   (b) Show that the union of a finite number of bounded subsets of X is bounded.

4. Let X be a set and let $p \in X$. Prove that $\mathcal{T} = \{X\} \cup \{U \subseteq X \mid p \notin U\}$ is a topology on X.

5. Show that a topological space X has the discrete topology if and only if its singleton sets $\{x\}$ are open.

6. Let $f : X \to Y$ be a continuous function of topological spaces. Show that, if $S \subseteq X$ is sequentially compact, then $f(S) \subseteq Y$ is sequentially compact.

7. Let X be a topological space with basis $\mathcal{B}$, and let S be a subset of X. Prove that the set $\mathcal{B}_S = \{S \cap B \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on S.

8. Let (X, d) be a metric space, and $S \subseteq X$ a subset. We now have two ways of defining a topology on S: we can restrict the metric d from X to S, and take the induced topology. Or, we can take the topology induced by d on X, and give S the subspace topology. Verify that these two topologies on S agree, so there is no ambiguity in how it should be topologized.

9. Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq B \subseteq X$. Prove that the subspace topology on A (as a subset of X) is the same as the subspace topology on A as a subset of B (with the subspace topology $\mathcal{T}_B$).

10. Let X be a topological space with the indiscrete topology.
    (a) Describe all closed subsets of X.
    (b) Suppose X contains more than one point. Show that X is not metrizable.
    (c) Show that X is compact.
    (d) Show that X is path-connected and connected.
    (e) Show that any sequence in X converges to every point of X. Conclude in particular that X is sequentially compact.
    (f) Let $A \subsetneq X$ be a proper subset. Show that the interior of A is $\varnothing$.
    (g) Let $A \subseteq X$ be a nonempty subset. Show that the closure of A is X.
    (h) Let $A \subseteq X$ be subset of X. When is it true that every point of X is an accumulation point of A? When is it true that every point of $X \setminus A$ is an accumulation point of A?

11. Recall that Sierpiński space $\mathbb{S}$ is the set $\mathbb{S} = \{0, 1\}$ with the topology $\Big\{\varnothing, \{0\}, \{0, 1\}\Big\}$.

(a) Show that $\mathbb{S}$ is not Hausdorff.

(b) Show that every continuous function $\mathbb{S} \to \mathbb{R}$ (with the standard topology) is constant.

(c) There are 4 possible functions $\mathbb{S} \to \mathbb{S}$. Determine which of these maps are continuous, and which are not continous. Which are homeomorphisms?

(d) Show that $\mathbb{S}$ is path-connected and connected.

(e) Show that $\mathbb{S}$ and all of its subsets are compact.

(f) Show that every sequence in $\mathbb{S}$ converges to 1. Under what conditions will a sequence converge to 0?

(g) Find all possible bases for $\mathbb{S}$.

(h) Let $(X, \mathcal{T})$ be a topological space. Show that $U \subseteq X$ is open if and only if the following map is continuous.

$$\chi_U : X \longrightarrow \mathbb{S}$$

$$\chi_U(x) = \begin{cases} 0, & x \in U \\ 1, & x \notin U. \end{cases}$$

12. Let A, B be subsets of a topological space X. Show that $\text{Int}(A) \cup \text{Int}(B) \subseteq \text{Int}(A \cup B)$, but that equality may not hold in general.

13. Consider the following topologies on $\mathbb{R}$.

- The topology induced by the Euclidean metric
- $\mathcal{T} = \{\mathbb{R}, \varnothing\}$
- $\mathcal{T} = \{\mathbb{R}, (0,1), \varnothing\}$
- $\mathcal{T} = \{\mathbb{R}, \{0,1\}, \{0\}, \{1\}, \varnothing\}$
- $\mathcal{T} = \{A \mid A \subseteq \mathbb{R}\}$
- $\mathcal{T} = \{A \mid A \subseteq \mathbb{R}, \mathbb{R} \setminus A \text{ is finite}\} \cup \{\varnothing\}$
- $\mathcal{T} = \{(-\infty, a) \mid a \in \mathbb{R}\} \cup \{\varnothing\} \cup \{\mathbb{R}\}$
- $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\varnothing\} \cup \{\mathbb{R}\}$
- $\mathcal{T} = \{A \mid A \subseteq \mathbb{R},\ 0 \in A\} \cup \{\varnothing\}$
- $\mathcal{T} = \{A \mid A \subseteq \mathbb{R},\ 0 \notin A\} \cup \{\mathbb{R}\}$
- $\mathcal{T} = \{A \mid A \subseteq \mathbb{R},\ 1 \in A\} \cup \{\varnothing\}$
- $\mathcal{T} = \{A \mid A \text{ is a union of intervals of the form } [a,b) \text{ for } a, b \in \mathbb{R}\ \} \cup \{\varnothing\}$

(a) For each topology, think about what convergence means for a sequence of real numbers. Write down some sequences and determine which (if any) real numbers they converge to.

(b) Write down some subsets of $\mathbb{R}$. For each topology, determine each subset's interior, closure, boundary, and accumulation points.

(c) Write down some subsets of $\mathbb{R}$. For each topology, determine whether the subset is T_1, Hausdorff, compact, connected, or path-connected.

14. Let $f : X \to Y$ be a function of topological spaces. Suppose that X can be written as a union of **open** subsets $X = \bigcup_{i \in I} U_i$. Suppose moreover that for each $i \in I$, the restriction $f|_{U_i} : U_i \to Y$ of f to U_i is continuous with respect to the subspace topology on U_i. Show that f is continuous.

15. Let $f, g : X \to \mathbb{R}$ be continuous functions.

(a) Show that the set $\{x \in X \mid f(x) \leq g(x)\}$ is closed.

(b) Show that the "minimum" function $m(x)$ is continuous:

$$m : X \to \mathbb{R}$$
$$m(x) = \min\{f(x), g(x)\}.$$

16. Let X be a topological space with basis $\mathcal{B}$.
 (a) Let $U \subseteq X$. Show that U is open if and only if, for each $u \in U$, there is some $B \in \mathcal{B}$ with $u \in B \subseteq U$.
 (b) Let $A \subseteq X$. Show that $a \in \text{Int}(A)$ if and only if there is some $B \in \mathcal{B}$ with $a \in B \subseteq A$.

17. (a) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $f : X \to Y$ be a function. Recall that the *graph* of f is defined to be the subset of $X \times Y$

$$\{ (x, f(x)) \in X \times Y \mid x \in X \}.$$

 Suppose that Y is Hausdorff. Show that, if f is continuous, then the graph of f is a closed subset of $X \times Y$ with respect to the product topology $\mathcal{T}_{X \times Y}$.
 (b) Find a counterexample when Y is not Hausdorff.

18. Let $A \subseteq X$ and $B \subseteq Y$ be subsets of topological spaces X and Y respectively. Show that $\overline{A} \times \overline{B} = \overline{A \times B}$ as subsets of $X \times Y$ with the product topology.

19. Let X and Y be Hausdorff topological spaces. Prove that the product $X \times Y$ (with the product topology) is Hausdorff.

20. Let X, Y, Z be topological spaces, and endow $X \times Y$ with the product topology. Let f be a function $f : Z \to X \times Y$, so f has the form

$$f : Z \to X \times Y$$
$$f(z) = (f_X(z), f_Y(z))$$

 for coordinate functions $f_X : Z \to X$ and $f_Y : Z \to Y$. Show that f is continuous if and only if its coordinate functions f_X and f_Y are continuous.

21. **Definition (Continuity in each variable).** Let X, Y, Z be topological spaces, and $X \times Y$ the topological space with the product topology. Let $F : X \times Y \to Z$ be a function. Then F is *continuous in each variable separately* if for each $y_0 \in Y$, and for each $x_0 \in X$, the following maps are continuous.

$$X \longrightarrow Z \qquad\qquad Y \longrightarrow Z$$
$$x \longmapsto F(x, y_0) \qquad\qquad y \longmapsto F(x_0, y).$$

 (a) Show that, if F is continuous, then it is continuous in each variable.
 (b) Show that the converse is false. *Hint:* Consider the function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$F(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0), \end{cases}$$

 and use the following result from real analysis.

Lemma. Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function, and fix a point (x_0, y_0) in $\mathbb{R} \times \mathbb{R}$. If F is continuous at (x_0, y_0), then for any parameterized line

$$x(t) = x_0 + at, \qquad y(t) = y_0 + bt \qquad (a, b \in \mathbb{R} \text{ any constants}),$$

the limit $\lim_{t \to 0} F\big(x(t), y(t)\big)$ exists and equals $F(x_0, y_0)$.

22. Let X be a topological space.
    (a) Suppose that X is Hausdorff. Let $x \in X$. Show that the intersection of all open sets containing x is equal to $\{x\}$.
    (b) Show that the converse statement does not hold. *Hint:* Consider ($\mathbb{R}$, cofinite).

23. Let X be a topological space, and let $A, B \subseteq X$. Then A and B form a separation of X if and only if they are disjoint nonempty sets such that $A \cup B = X$ and $\overline{A} \cap B = A \cap \overline{B} = \varnothing$.

24. Let X be a topological space, and let $\{C_i\}_{i \in I}$ be a (nonempty) collection of connected subsets of X. Suppose that, for some fixed $j \in I$, the intersection $C_i \cap C_j \neq \varnothing$ for all $i \in I$. Prove that $\bigcup_{i \in I} C_i$ is connected.

25. Let X be a topological space.
    (a) Suppose that $X = U \cup V$ is a separation of X. Prove or disprove: U and V is a union of connected components of X.
    (b) Suppose that $X = U \cup V$ is a decomposition of X into two nonempty disjoint subsets, each of which is a union of conneccted components of X. Prove or disprove: U and V are a separation of X.

26. **Definition (Totally disconnected space).** A topological space X is called *totally disconnected* if its connected components are all singletons $\{x\}$.
    (a) Let X be a topological space with the discrete topology. Show that X is totally disconnected.
    (b) Find an example of a topological space X that is totally disconnected, but not discrete.

27. Determine whether the set $\mathbb{R}^2 \setminus \mathbb{Q}^2$ is connected or path-connected.

28. Let $(X, \mathcal{T}_X)$ be a compact topological space, and let $(Y, \mathcal{T}_Y)$ be a Hausdorff topological space. Let $f : X \to Y$ be a continuous map. Show that f is a *closed map*, that is, $f(C) \subseteq Y$ is closed whenever $C \subseteq X$ is closed.

29. **Definition (Dense subsets; nowhere dense subsets).** Let $(X, \mathcal{T})$ be a topological space. A subset $A \subseteq X$ is called *dense* if $\overline{A} = X$. The subset A is called *nowhere dense* if the interior of $\overline{A}$ is empty.
    (a) Give an example of a subset of $\mathbb{R}$ that is dense, and a subset of $\mathbb{R}$ that is nowhere dense. Give an example of a set that is neither. Can a set be both dense and nowhere dense?
    (b) Let A be a dense subset of a space X. Show that any open subset $U \subseteq X$ satisfies $\overline{U \cap A} = \overline{U}$.
    (c) Show that a subset $A \subseteq X$ of a space X is nowhere dense if and only if $X \setminus \overline{A}$ is a dense open subset of X.

(d) Let $f : X \to Y$ be a contitnuous function of topological spaces, and let $A \subseteq X$ be a dense subset. Suppose that Y is Hausdorff. Explain why a continuous map $f : X \to Y$ is completely determined by its values on A.

30. (a) Let X be a topological space, and $A \subseteq X$. Show that, if A has no accumulation points, then A is closed.

    (b) Prove the following.

    Theorem. Let X be a topological space. If X is compact, then every infinite subset S of X has an accumulation point.

    Hint: First show that $\{(X \setminus S) \cup \{s\}\}_{s \in S}$ is an open cover of X.

31. Let $K = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Let $\mathbb{R}_K$ denote the set $\mathbb{R}$ with the topology defined by the basis

$$\mathcal{B} = \{(a,b) \mid a,b \in \mathbb{R}\} \cup \{(a,b) \setminus ((a,b) \cap K) \mid a,b \in \mathbb{R}\},$$

called the *K-topology.* The space $\mathbb{R}_K$ is a useful source of counterexamples in point-set topology.

(a) Verify that $\mathcal{B}$ really is a basis, so it generates a well-defined topology.

(b) Explain why any set that is open in the standard topology on $\mathbb{R}$ is also open in $\mathbb{R}_K$. This is the statement that the topology on $\mathbb{R}_K$ is *finer* than the topology on $\mathbb{R}$.

(c) Show that $\mathbb{R}_K$ is Hausdorff (and therefore also T_1).

(d) Show that the set K is closed in $\mathbb{R}_K$.

(e) What are the limits of the sequence $\left(\frac{1}{n}\right)_{n \in \mathbb{N}}$ in $\mathbb{R}_K$?

(f) What are the accumulation points of the set $K \subseteq \mathbb{R}_K$?

(g) Prove that $[0,1] \subseteq \mathbb{R}_K$ is not compact. *Hint:* Problem 30.