

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let $X = \{a, b, c, d\}$ with the topology

$$\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}.$$

- (a) Is X a T_1 -space?
 - (b) Is X Hausdorff?
 - (c) Find the interior, closure, and boundary of $\{a, c, d\}$.
 - (d) Find the interior, closure, and boundary of $\{a, b\}$.
2. (a) Let (X, \mathcal{T}) be a topological space, and let $I : X \rightarrow X$ be the identity function, defined by $I(x) = x$ for all $x \in X$. Show that I is always continuous when the domain and codomain are given the same topology.
(b) Now consider $I : \mathbb{R} \rightarrow \mathbb{R}$, and investigate whether I is continuous when we allow the domain and codomain to carry different topologies – the answer now depends on the two topologies chosen. Consider (for example) the discrete topology, the indiscrete topology, the cofinite topology, and the Euclidean topology.
 3. Let (X, \mathcal{T}) be a topological space.
 - (a) Show that \mathcal{T} is a basis for \mathcal{T} .
 - (b) Suppose that \mathcal{B} is a basis for \mathcal{T} . Show that any collection of open sets in X containing \mathcal{B} is also a basis for \mathcal{T} .
 4. Verify that the set of open intervals $(a, b) \subseteq \mathbb{R}$ is a basis for \mathbb{R} (with the Euclidean topology).
 5. Let X, Y, Z be sets and let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be functions.
 - (a) Show that if f and g are both surjective, then $g \circ f$ is surjective.
 - (b) Show that if f and g are both injective, then $g \circ f$ is injective.
 - (c) Show by example that, if we only assume one of f and g is surjective, then $g \circ f$ need not be surjective.
 - (d) Show by example that, if we only assume one of f and g is injective, then $g \circ f$ need not be injective.

Worksheet problems

(Hand these questions in!)

- Worksheet #11 Problems 2(a), 3.
- Worksheet #12 Problems 1, 4, 5.

Assignment questions

(Hand these questions in!)

- Let (X, \mathcal{T}_X) be a topological space, and let $A, B \subseteq X$.
 - Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
 - Show by example that $\overline{A \cap B}$ need not equal $\overline{A} \cap \overline{B}$.
- Let (X, \mathcal{T}_X) be a topological space, and suppose $X = A \cup B$ for two **closed** subsets $A, B \subseteq X$. Let (Y, \mathcal{T}_Y) be a topological space and $f : X \rightarrow Y$ a function. Show that, if $f|_A$ and $f|_B$ are continuous (with respect to the subspace topologies on A and B), then f is continuous.
- (a) Suppose that (X, \mathcal{T}_X) is a topological space, and that (Y, \mathcal{T}_Y) is a **Hausdorff** topological space. Let $f : X \rightarrow Y$ and $g : X \rightarrow Y$ be continuous functions. Suppose that $A \subseteq X$ is a subset such that

$$f(a) = g(a) \quad \text{for all } a \in A.$$

Prove that

$$f(x) = g(x) \quad \text{for all } x \in \overline{A}.$$

This says that the values of a continuous function on \overline{A} are completely determined by its values on A .

- Suppose you have a function of topological spaces $f : \mathbb{R} \rightarrow Y$, where \mathbb{R} has the standard topology and Y is Hausdorff. Briefly explain why the previous problem implies, if f is continuous, then it is completely determined by its values on \mathbb{Q} .
- (a) Show that the continuous image of a Hausdorff topological space need not be Hausdorff. Specifically, find topological spaces (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) such that X is Hausdorff but $f(X)$ (as a subspace of Y with the subspace topology) is not Hausdorff.
 - Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces, and assume that (Y, \mathcal{T}_Y) is Hausdorff. Show that, if there exists a continuous **injective** map $f : X \rightarrow Y$, then X must also be Hausdorff.
 - Let (X, \mathcal{T}) be a topological space, and let \mathcal{B} be a basis for \mathcal{T} . For a subset $A \subseteq X$, prove the following.
 - $\text{Int}(A) = \{a \in A \mid \text{there exists a basis element } B \text{ with } a \in B \subseteq A\}$
 - $\overline{A} = \{x \in X \mid \text{any basis element } B \in \mathcal{B} \text{ containing } x \text{ must contain a point of } A\}$