Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let $X = \{a, b, c, d\}$. Let $\mathcal{T}$ be the topology on X

$$\mathcal{T} = \{\varnothing, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}.$$

   (a) Which elements of X are limits of the constant sequence $x_n = d$? The constant sequence $x_n = a$? The constant sequence $x_n = b$?

   (b) Give an example of a sequence in X that does not converge.

2. Let $X = \{0, 1\}$. Find a topology on X for which the following sequence converges:

$$0\ 1\ 0\ 1\ 0\ 1\ 0\ 1 \cdots$$

3. Suppose that $(X, \mathcal{T})$ is a topological space, and that $(a_n)_{n\in\mathbb{N}}$ is a sequence in X that converges to $a_\infty \in X$. Prove that any subsequence of $(a_n)_{n\in\mathbb{N}}$ converges to $a_\infty \in X$.

4. Let X be a topological space with the indiscrete topology. Prove that any sequence of points in X converges to every point in X.

Worksheet problems

(Hand these questions in!)

- Worksheet #13 Problems 1, 3.
- Worksheet #14 Problems 2(b), 4.

Assignment questions

(Hand these questions in!)

1. (a) Let A be a subset of a topological space X. Show that $\partial(\text{Int}(A))$ is contained in ∂A, but show by example that these sets need not be equal.

   (b) Let A be a subset of a topological space X. Show that $\partial(\overline{A})$ is contained in ∂A, but show by example that these sets need not be equal.

2. **Definition (path).** Let X be a topological space, and let $I = [0, 1] \subseteq \mathbb{R}$ be the unit interval with the standard topology. A *path* in X is a continuous function $\gamma : I \to X$. For points $x, y \in X$, we say that a path γ is a *path from x to y* if $\gamma(0) = x$ and $\gamma(1) = y$.

   Note that the path γ does not need to be injective. For example, the constant path $\gamma(t) = x$ is a path from x to x.
   For these problems (as always) you may assume any standard results about which functions between Euclidean spaces (with the standard topology) are continuous.

(a) Given two points $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, construct a path from x to y.

(b) Let x, y be points in the space $\mathbb{R}$ with the cofinite topology. Construct (with proof) a path from x to y.

(c) Let $X = \{a, b, c, d\}$ be a topological space with the topology

$$\mathcal{T} = \Big\{ \varnothing, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}, \{a,b,c,d\} \Big\}.$$

Construct (with proof) a path in X from a to d.

3. Let X be a topological space.

   (a) Let $x, y \in X$ and suppose that there exists a path from x to y. Show that there exists a path from y to x.

   (b) Let $x, y, z \in X$. Show that, if there exists a path from x to y, and a path from y to z, then there exists a path from x to z. *Hint:* Homework #10 Problem 2.

   Remark: Although we will not formally define this term, we remark that this problem shows that, for a topological space X, the condition "there exists a path from x to y" defines an *equivalence relation* on the points of X.

4. Let $(X, \mathcal{T}_X)$ be a topological space, and endow the product $X \times X$ with the product topology $\mathcal{T}_{X \times X}$. The set

$$\Delta = \{ \ (x,x) \mid x \in X \ \} \ \subseteq \ X \times X$$

   is called the *diagonal* of $X \times X$. Prove that X is Hausdorff if and only if the diagonal Δ is a closed subset of $X \times X$.

5. Consider the set $\mathbb{R}$ with the **cofinite** topology. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of points in $\mathbb{R}$.

   (a) Suppose that the sequence has the property that each term is repeated at most finitely many times. More precisely, suppose for each $r \in \mathbb{R}$ that $r = a_n$ for at most finitely many values of $n \in \mathbb{N}$. Which points of $\mathbb{R}$ are limits of the sequence $(a_n)_{n \in \mathbb{N}}$?

   (b) Now suppose the set $\{a_n \mid n \in \mathbb{N}\}$ is finite. Under what conditions will the sequence converge, and what will its limit(s) be?

   Remember to justify your solutions!