Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let $(X, \mathcal{T})$ be a topological space. Show that any subset $A = \{x\} \subseteq X$ of a single element is connected.

2. Let $X = \{a, b, c, d\}$ with the topology
$$\mathcal{T} = \{\varnothing, \{a\}, \{a, b\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}.$$
Is X connected?

3. (a) Show that, for $a, b \in \mathbb{R}$, the subsets $\varnothing, \{a\}, (a, b), (a, b], [a, b), [a, b], (a, \infty), [a, \infty), (\infty, b), (\infty, b]$, and $\mathbb{R}$ of $\mathbb{R}$ are all intervals in the sense of Problem b.
   (b) Show that every interval must have one of these forms.

4. Give an example of a subset A of $\mathbb{R}$ (with the standard topology) such that A is not connected, but $\overline{A}$ is connected. (Compare to Assignment Problem 1)

Worksheet problems

(Hand these questions in!)

- Worksheet #15 Problems 3, 5, 6(b).

Assignment questions

(Hand these questions in!)

1. Let $(X, \mathcal{T}_X)$ be a topological space, and let $A \subseteq X$ be a connected subset. Let B be any subset such that $A \subseteq B \subseteq \overline{A}$. Prove that B is connected.
   Remark: This shows in particular that if A is connected, then so is $\overline{A}$.

2. (a) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $f : X \to Y$ be a continuous map. Prove that if X is connected, then $f(X)$ is connected. In other words, the continuous image of a connected space is connected.
   (b) Recall the Intermediate Value Theorem from real analysis (which you may use without proof).

      Intermediate Value Theorem. If $f : [a, b] \to \mathbb{R}$ is continuous and d lies between $f(a)$ and $f(b)$ (i.e. either $f(a) \le d \le f(b)$ or $f(b) \le d \le f(a)$), then there exists $c \in [a, b]$ such that $f(c) = d$.

   Define a subset $A \subseteq \mathbb{R}$ to be an *interval* if whenever $x, y \in A$ and z lies between x and y, then $z \in A$.
   Prove that any interval of $\mathbb{R}$ is connected. *Hint:* Worksheet #15 Problem 4.
   (c) Prove that any subset of $\mathbb{R}$ that is not an interval is disconnected.

These last two results together prove:

Theorem (Connected subsets of $\mathbb{R}$). A subset of $\mathbb{R}$ is a connected if and only if it is an interval.

3. (a) Prove the following result.

   Theorem (Generalized Intermediate Value Theorem). Let $(X, \mathcal{T}_X)$ be a connected topological space, and let $f : X \to \mathbb{R}$ be a continuous function (where the topology on $\mathbb{R}$ is induced by the Euclidean metric). If $x, y \in X$ and c lies between $f(x)$ and $f(y)$, then there exists $z \in X$ such that $f(z) = c$.

   (b) Prove that any continuous function $f : [0,1] \to [0,1]$ has a fixed point. (In other words, show that there is some $x \in [0,1]$ so that $f(x) = x$).
   Hint: Consider the function

$$\begin{aligned} g &: [0,1] \to \mathbb{R} \\ g(x) &= f(x) - x. \end{aligned}$$

4. **Definition (Connected components of a topological space).** Let $(X, \mathcal{T}_X)$ be a topological space. A subset $C \subseteq X$ is called a *connected component* of X if

   (i) C is connected;

   (ii) if C is contained in a connected subset A, then $C = A$.

   In other words, the connected components are the 'maximal' connected subsets of X.

   (a) Show that any connected component of X is closed. (*Hint*: Problem 1).

   (b) Let $x \in X$. Show that the set

$$\bigcup_{\substack{A \text{ is a connected set,} \\ x \in A}} A$$

   is a connected component of X.

   (c) Show that X is the **disjoint union** of its connected components. In other words, show that every point of X is contained in one, and only one, connected component.

   (d) Determine the connected components of $\mathbb{Q}$ (with the Euclidean metric). (Remember to rigorously justify your answer!)

   (e) Deduce from the example of $\mathbb{Q}$ that connected components need not be open.

   (f) Suppose that X has the property that every point has a connected neighbourhood. Show that the connected components of X are open.