Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. (a) Give an example of a metric space and a subset that is both open and closed. Give an example of a subset that is neither open nor closed.

   (b) Recite the Topologist Scout Oath:

   "On my honour, I will do my best
   to never claim to prove a set is closed by showing that it is not open,
   and to never claim to prove a set is open by showing that it is not closed."

2. Let X and Y be sets, and $f : X \to Y$ any function. Show that $f^{-1}(Y) = X$, and $f^{-1}(\varnothing) = \varnothing$.

3. Rigorously prove that the following functions $f : \mathbb{R} \to \mathbb{R}$ are continuous. (Here, $\mathbb{R}$ implicitly has the Euclidean metric.)

   (a) $f(x) = 5$ (b) $f(x) = 2x + 3$ (c) $f(x) = x^2$

   (d) $f(x) = g(x) + h(x)$, for continuous functions g and h.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^2 + 2$. Find the inverse images of the following sets, and verify that they are open.

   (a) $\mathbb{R}$ (b) $(-1, 1)$ (c) $(2, 3)$ (d) $(6, \infty)$

5. Let (X, d) be a metric spaces. Show that the identity function

$$\begin{aligned} g : X &\longrightarrow X \\ g(x) &= x \qquad \text{for all } x \in X \end{aligned}$$

   is always continuous.

6. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $y_0 \in Y$. Show that the constant function

$$\begin{aligned} f : X &\longrightarrow Y \\ f(x) &= y_0 \qquad \text{for all } x \in X \end{aligned}$$

   is always continuous.

7. See the definition of accumulation points and isolated points in Problem (4) below. Let $X = \mathbb{R}$. Find the set of accumulation points and the set of isolated points for each of the following subsets of X.

   (a) $S = \{0\}$ (b) $S = (0, 1)$ (c) $S = \mathbb{Q}$ (d) $S = \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\}$

Worksheet problems

(Hand these questions in!)

- Worksheet # 2 Problem 1(b), 1(d), and Problem 4
- Worksheet # 3 Problem 1

Assignment questions

(Hand these questions in!)

1. Let $f : X \to Y$ be a function of sets X and Y. Let $A \subseteq X$ and $C \subseteq Y$. For each of the following, determine whether you can replace the symbol $\square$ with $\subseteq, \supseteq, =$, or none of the above. Justify your answer by giving a proof of any set-containment or set-equality you claim. If set-equality does not hold in general, give a counterexample.

   (a) $A \quad \square \quad f^{-1}(f(A)))$ (b) $C \quad \square \quad f(f^{-1}(C)))$

2. Let $f : X \to Y$ be a function between metric spaces. We proved that a subset $S \subseteq X$ inherits a metric space structure from the metric on X. Recall that the *restriction* of f to S, often written $f|_S$, is the function

$$f|_S : S \longrightarrow Y$$
$$f|_S(s) = f(s).$$

   Prove that, if f is a continuous function, then its restriction $f|_S$ to S is also a continuous function.

3. Prove the following theorem.

   Theorem (Equivalent definition of continuity.) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a function. Then f is continuous if and only if it satisfies the following property: for every closed set $C \subseteq Y$, the preimage $f^{-1}(C)$ is closed.

4. Consider the following definition.

   Definition (Accumulation points of a set.) Let (X, d) be a metric space, and let $S \subseteq X$ be a set. A point $x \in X$ is called an *accumulation point* of S if for every $r > 0$ the ball $B_r(x)$ around x contains at least one point of S distinct from x. Note that x may or may not itself be an element of S.

   (a) An element $s \in S$ that is not an accumulation point of S is called an *isolated point* of S. Negate the definition of an accumulation point to give a precise statement of what it means to be an isolated point.

   (b) Prove that the following definition of accumulation point is equivalent to the one above. In other words, show that a point $x \in X$ is an accumulation point of a set $S \subseteq X$ if and only if it satisfies the following property.

      Alternative Definition (Accumulation points of a set.) Let (X, d) be a metric space, and let $S \subseteq X$ be a set. A point $x \in X$ is called an *accumulation point* of S if every open subset U of X containing x also contains a point in S distinct from x.

   (c) Let (X, d) be a metric space and let $S \subseteq X$ be a **closed** subset. Let x be an accumulation point of S. Show that x is contained in S.

   (d) Let (X, d) be a metric space and let $S \subseteq X$ be any subset. Let x be an accumulation point of S, and let $B_r(x)$ be a ball centered around x of some radius $r > 0$. Show that $B_r(x)$ contains infinitely many points of S.