Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of elements in a metric space X, and let $a_\infty \in X$. Show that the following statement is equivalent to the statement that $\lim_{n\to\infty} a_n = a_\infty$.

For every $\epsilon > 0$, the ball $B_{\epsilon}(a_{\infty})$ about a_{∞} contains a_n for all but finitely many n.

- 2. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of elements in a metric space X.
 - (a) Let $x \in X$. Negate the definition of convergence to state what it means for the sequence to **not** converge to x.
 - (b) Formally state what it means for the sequence $(a_n)_{n\in\mathbb{N}}$ to be non-convergent.
- 3. Rigorously determine the limits of the following sequences of real numbers, or prove that they do not converge.
 - (a) $a_n = 0$ (b) $a_n = \frac{1}{n^2}$ (c) $a_n = n$ (d) $a_n = (-1)^n$
- 4. Suppose that $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are sequences of real numbers that converge to a_∞ and b_∞ , respectively. Prove that the sequence $(a_n + b_n)_{n\in\mathbb{N}}$ converges to $(a_\infty + b_\infty)$.
- 5. Consider the sequence $\left(\frac{(-1)^n}{n}\right)_{n\in\mathbb{N}}$ in \mathbb{R} . Let $\epsilon>0$ be fixed. Find a number $N\in\mathbb{R}$ so that, for all $m,n\geq N$, $\left|\frac{(-1)^n}{n}-\frac{(-1)^m}{m}\right|<\epsilon.$

This shows that the sequence $\left(\frac{(-1)^n}{n}\right)_{n\in\mathbb{N}}$ is Cauchy (as defined in Question 5).

Worksheet Problems

(Hand these questions in!)

• Worksheet 4 Problem 1, 2

Assignment questions

(Hand these questions in!)

- 1. Let $f: X \to Y$ be a function of sets X and Y. Let $C, D \subseteq Y$. For each of the following, determine whether you can replace the symbol \square with $\subseteq, \supseteq, =$, or none of the above. Justify your answer by giving a proof of any set-containment or set-equality you claim. If set-equality does not hold in general, give a counterexample.
 - (a) $f^{-1}(C \cup D) \quad \Box \quad f^{-1}(C) \cup f^{-1}(D)$ (b) $f^{-1}(C \cap D) \quad \Box \quad f^{-1}(C) \cap f^{-1}(D)$
 - (c) For $C \subseteq D$, $f^{-1}(D \setminus C) \square f^{-1}(D) \setminus f^{-1}(C)$

2. **Definition (The discrete metric.)** Given a set X, the discrete metric on X is the metric $d_X: X \times X \to \mathbb{R}$ defined by

$$d_X(x,x') = \begin{cases} 0, & x = x' \\ 1, & x \neq x' \end{cases}$$
 for all $x, x' \in X$.

Let (X, d) be a metric space with the discrete metric.

- (a) Show that every subset of X is both open and closed.
- (b) Let (Y, d_Y) be any metric space. Prove that **every** function $f: X \to Y$ is continuous.
- 3. In this question, we will prove the following result.

Theorem (Another characterization of closed subsets). Let (X, d) be a metric space, and let $A \subseteq X$. Then A is closed if and only if it satisfies the following condition: If $(a_n)_{n \in \mathbb{N}}$ is a convergent sequence of points in A converging to a point $a_{\infty} \in X$, then the limit a_{∞} is contained in A.

- (a) Suppose that $A \subseteq X$ is closed. Let a_{∞} be the limit of a convergent sequence $(a_n)_{n \in \mathbb{N}}$ of points in A. Show that $a_{\infty} \in A$.
- (b) Suppose that $A \subseteq X$ is a subset that contains the limits of every one of its convergent sequences. Prove that A is closed.
- 4. Prove the following result:

Theorem (Another definition of continuous functions.) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f: X \to Y$ be a function. Then f is continuous if and only if the following condition holds: given any convergent sequence $(a_n)_{n\in\mathbb{N}}$ in X, then $(f(a_n))_{n\in\mathbb{N}}$ converges in Y, and

$$\lim_{n \to \infty} f(a_n) = f\left(\lim_{n \to \infty} a_n\right).$$

5. Consider the following definition.

Definition (Cauchy sequence.) Let (X,d) be a metric space. Then a sequence $(a_n)_{n\in\mathbb{N}}$ of points in X is called a *Cauchy sequence* if for every $\epsilon>0$ there exists some $N\in\mathbb{R}$ such that $d(a_n,a_m)<\epsilon$ whenever $n,m\geq N$.

- (a) Prove that every convergent sequence in X is a Cauchy sequence.
- (b) Give an example of a metric space (X, d) and a sequence $(a_n)_{n \in \mathbb{N}}$ in X that is Cauchy but does not converge. Fully justify your solution!