Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Consider $\mathbb{R}$ with the Euclidean metric. Which of the following maps $f : \mathbb{R} \to \mathbb{R}$ are homeomorphisms (Assignment Question 2)?

   (a) $f(x) = ax + b$ (b) $f(x) = x^2$ (c) $f(x) = x^3$ (d) $f(x) = \sin(x)$

2. Let $f : X \to Y$ be an **invertible** function of sets, and let $S \subseteq Y$. The notation $f^{-1}(S)$ could denote either the preimage of S under f, or the image of S under the inverse function f^{-1}. Show that these two sets are equal, so there is no ambiguity in using the notation $f^{-1}(S)$.

3. Let $f : X \to Y$ be an **invertible** function of sets.
   (a) Show that, for subsets $B \subseteq Y$, there is equality $f(f^{-1}(B)) = B$.
   (b) Show that, for subsets $A \subseteq X$, there is equality $f^{-1}(f(A)) = A$.

4. See the definition of bounded in Assignment Question 3.
   (a) Negate the definition of *bounded* to state what it means for a subset S of a metric space to be *unbounded*.
   (b) Is $\varnothing$ a bounded set?
   (c) Show that any **finite** subset of a metric space is bounded.

5. Give examples of subsets of $\mathbb{R}$ (with the Euclidean metric) that satsify the following.

   (a) open, and bounded
   (b) closed, and bounded
   (c) open, and unbounded
   (d) closed, and unbounded

6. Consider $\mathbb{R}$ with the Euclidean metric. For each of the following sets A, find $\mathrm{Int}(A)$, $\overline{A}$, ∂A, $\mathrm{Int}(\mathbb{R} \setminus A)$, $\overline{\mathbb{R} \setminus A}$, and $\partial(\mathbb{R} \setminus A)$. See Assignment Question 4 for the definition of a boundary ∂.

   (a) $\mathbb{R}$
   (b) $[0, 1]$
   (c) $(0, 1)$
   (d) $(0, 1]$
   (e) $\{0, 1\}$
   (f) $\{\frac{1}{n} \mid n \in \mathbb{N}\}$

7. Consider the real numbers $\mathbb{R}$ with the Euclidean metric. Find examples of subsets A of $\mathbb{R}$ with the following properties.
   (a) $\partial(A) = \varnothing$
   (b) A has a nonempty boundary, and A contains its boundary ∂A.
   (c) A has a nonempty boundary, and A contains no points in its boundary
   (d) A has a nonempty boundary, and A contains some but not all of the points in its boundary.
   (e) A has a nonempty boundary, and $A = \partial A$.
   (f) A is a **proper** subset of ∂A.

8. Let X be a nonempty set with the discrete metric. Let $A \subseteq X$. Show that $A = \mathrm{Int}(A) = \overline{A}$. Conclude that $\partial A = \varnothing$.

Worksheet problems

(Hand these questions in!)

- Worksheet #5 Problems 1(a), 2, 3.

Assignment questions

(Hand these questions in!)

1. Let $f : X \to Y$ be a function of sets X and Y. Let $A, B \subseteq X$. For each of the following, determine whether you can replace the symbol $\square$ with $\subseteq, \supseteq, =$, or none of the above. Justify your answer by giving a proof of any set-containment or set-equality you claim. If set-equality does not hold in general, give a counterexample.

   (a) $f(A \cap B) \quad \square \quad f(A) \cap f(B)$

   (b) $f(A \cup B) \quad \square \quad f(A) \cup f(B)$

   (c) For $A \subseteq B$, $\quad f(B \setminus A) \quad \square \quad f(B) \setminus f(A)$

2. Consider the following definition.

   Definition (Homeomorphism.) Let X and Y be metric spaces. Then a map $f : X \to Y$ is a *homeomorphism* if
   - f is continuous;
   - f has an inverse f^{-1};
   - f^{-1} is continuous.

   The metric space X is called *homeomorphic* to Y if there exists a homeomorphism $f : X \to Y$.

   (a) Show that, if $f : X \to Y$ is a homeomorphism, then $f^{-1} : Y \to X$ is a homeomorphism. Conclude that X is homeomorphic to Y if and only if Y is homeomorphic to X. (We simply call the two spaces *homeomorphic*.)

   (b) Let $f : X \to Y$ be a homeomorphism of metric spaces. Show that

   $$\begin{aligned} \{U \subseteq Y \mid U \text{ is open}\} &\longrightarrow \{V \subseteq X \mid V \text{ is open}\} \\ U &\longmapsto f^{-1}(U) \end{aligned}$$

   defines a bijection between the collection of all open subsets of Y, and the collection of all open subsets of X.
   Hint: To show it is a bijection, you could check that the assignment $V \mapsto f(V)$ is its inverse.

   (c) Consider the function

   $$\begin{aligned} f : (\mathbb{R}, \text{ discrete metric}) &\to (\mathbb{R}, \text{ Euclidean metric}) \\ f(x) &= x \end{aligned}$$

   Prove that f is continuous and invertible, but its inverse f^{-1} is not continuous.

Remark: Note the contrast to other mathematical fields, such as linear algebra: if a linear map has an inverse, then the inverse is automatically linear. This exercise shows that this is not true for continuous maps!

3. Consider the following definition.

   Definition (Bounded subset.) Let (X, d) be a metric space. A subset $S \subseteq X$ is called *bounded* if there is some $x_0 \in X$ and some $R \in \mathbb{R}$ with $R > 0$ such that $S \subseteq B_R(x_0)$.

   (a) Let (X, d) be a metric space, and $A \subseteq X$ a subset. Show that A is bounded if and only if, for every $x \in A$, there is some $R_x > 0$ such that $A \subseteq B_{R_x}(x)$.

   (b) Let (X, d) be a metric space. Suppose that $(a_n)_{n \in \mathbb{N}}$ is a sequence in X converging to an element a_∞. Show that the set $\{ a_n \mid n \in \mathbb{N} \}$ is a bounded subset of X.

4. Consider the following definition.

   Definition (Boundary of a set A.) Let (X, d) be a metric space, and let $A \subseteq X$. Then the *boundary* of A, denoted ∂A, is the set $\overline{A} \setminus \operatorname{Int}(A)$.

   Let (X, d) be a metric space, and let $A \subseteq X$.

   (a) Prove that $\operatorname{Int}(A) = \overline{A} \setminus \partial A$.

   (b) Prove that $\partial A = \overline{A} \cap \overline{(X \setminus A)}$.

   (c) Conclude from part (b) that ∂A is closed.

   (d) Additionally conclude from part (b) that $\partial A = \partial(X \setminus A)$.

   (e) Prove the following characteriziation of points in the boundary:

   Theorem (An equivalent definition of ∂A). Let (X, d) be a metric space, and let $A \subseteq X$. Then $x \in \partial A$ if and only if every ball $B_r(x)$ about x contains at least one point of A, and at least one point of $X \setminus A$.

   (f) Deduce that we can classify every point of X in one of three mutually exclusive categories:

   (i) interior points of A;

   (ii) interior points of $X \setminus A$;

   (iii) points in the (common) boundary of A and $X \setminus A$.