Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Consider the sequence of real numbers $1, 2, 3, 4, 5, \ldots$. Which of the following are subsequences?

   (a) $1, 1, 2, 2, 3, 3, 4, 4, 5, 5, \ldots$

   (b) $1, 1, 1, 1, 1, 1, 1, \ldots$

   (c) $2, 4, 6, 8, 10, 12, \ldots$

   (d) $1, 3, 2, 4, 5, 7, 6, 8, 9, 11, \ldots$

2. Let (X, d) be a metric space, and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points in X. Recall that we proved that, if $\lim\limits_{n\to\infty} a_n = a_\infty$, then any subsequence of $(a_n)_{n\in\mathbb{N}}$ also converges to a_∞.

   (a) Suppose that $(a_n)_{n\in\mathbb{N}}$ has a subsequence that does not converge. Prove that $(a_n)_{n\in\mathbb{N}}$ does not converge.

   (b) Suppose that $(a_n)_{n\in\mathbb{N}}$ has a subsequence converging to $a \in X$, and a different subsequence converging to $b \in X$, with $a \neq b$. Prove that $(a_n)_{n\in\mathbb{N}}$ does not converge.

3. Let (X, d) be a metric space, and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points in X. Show that, if $\{a_n \mid n \in \mathbb{N}\}$ is a finite set, then $(a_n)_{n\in\mathbb{N}}$ must have a subsequence that is constant (and, in particular, convergent).

4. Let (X, d) be a metric space, and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points in X. Suppose that the set $\{a_n \mid n \in \mathbb{N}\}$ is unbounded. Explain why $(a_n)_{n\in\mathbb{N}}$ cannot converge.

5. Find examples of sequences $(a_n)_{n\in\mathbb{N}}$ of real numbers with the following properties.

   (a) $\{a_n \mid n \in \mathbb{N}\}$ is unbounded, but $(a_n)_{n\in\mathbb{N}}$ has a convergent subsequence

   (b) $(a_n)_{n\in\mathbb{N}}$ has no convergent subsequences

   (c) $(a_n)_{n\in\mathbb{N}}$ is not an increasing sequence, but it has an increasing subsequence

   (d) $(a_n)_{n\in\mathbb{N}}$ has four subsequences that each converge to a distinct limit point

6. Write down a bounded sequence in $\mathbb{R}$. Can you identify a convergent subsequence?

7. Determine which of the following subsets of of $\mathbb{R}^2$ can be expressed as the Cartesian product of two subsets of $\mathbb{R}$.

   (a) $\{(x, y) \mid x \in \mathbb{Q}\}$

   (b) $\{(x, y) \mid x, y \in \mathbb{Q}\}$

   (c) $\{(x, y) \mid x > y\}$

   (d) $\{(x, y) \mid 0 < y \leq 1\}$

   (e) $\{(x, y) \mid x^2 + y^2 < 1\}$

   (f) $\{(x, y) \mid x^2 + y^2 = 1\}$

   (g) $\{(x, y) \mid x = 3\}$

   (h) $\{(x, y) \mid x + y = 3\}$

8. (a) For which values of r is the square $(-r, r) \times (-r, r) \subseteq \mathbb{R}^2$ contained in the unit ball $\{(x, y) \mid x^2 + y^2 < 1\}$?

   (b) For which values of r is the r-ball $\{(x, y) \mid x^2 + y^2 < r^2\}$ contained in the square $(-1, 1) \times (-1, 1) \subseteq \mathbb{R}^2$?

Worksheet problems

(Hand these questions in!)

- Worksheet #6, Problem 2

Assignment questions

(Hand these questions in!)

1. For sets X and Y, let $A, B \subseteq X$ and $C, D \subseteq Y$. Consider the Cartesian product $X \times Y$. For each of the following, determine whether you can replace the symbol $\square$ with $\subseteq, \supseteq, =$, or none of the above. Justify your answer by giving a proof of any set-containment or set-equality you claim. If set-equality does not hold in general, give a counterexample.
   (a) $(A \times C) \cup (B \times D) \quad \square \quad (A \cup B) \times (C \cup D)$
   (b) $(A \times C) \cap (B \times D) \quad \square \quad (A \cap B) \times (C \cap D)$
   (c) $(X \setminus A) \times (Y \setminus C) \quad \square \quad (X \times Y) \setminus (A \times C)$

2. Consider the real numbers $\mathbb{R}$ with the Euclidean metric. Determine the interior, closure, and boundary of the subset $\mathbb{Q} \subseteq \mathbb{R}$. Remember to rigorously justify your solution!

3. Let (X, d) be a metric space, and $(a_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in X. Show that $\{a_n \mid n \in \mathbb{N}\}$ is a bounded subset of X.

4. Recall the following result from real analysis (which you do not need to prove):

   Theorem (Bolzano–Weierstrass). Let $A \subseteq \mathbb{R}^n$ be a bounded infinite set. Then A has an accumulation point $x \in \mathbb{R}^n$.

   Prove the following result (which is sometimes also called the Bolzano–Weierstrass theorem):

   Theorem (Sequential compactness in $\mathbb{R}^n$). Consider the space $\mathbb{R}^n$ with the Euclidean metric. Let $S \subseteq \mathbb{R}^n$ be a subset. Then S is sequentially compact if and only if S is closed and bounded.

5. (a) **Definition (Open cover).** A collection $\{U_i\}_{i \in I}$ of open subsets of a metric space X is an *open cover* of X if $X = \bigcup_{i \in I} U_i$. In other words, every point in X lies in some set U_i.

   Suppose that (X, d) is a sequentially compact metric space. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. Prove that (associated to the open cover $\mathcal{U}$) there exists a real number $\delta > 0$ with the following property: for every $x \in X$, there is some associated index $i_x \in I$ such that $B_\delta(x) \subseteq U_{i_x}$.

   (b) **Definition (ϵ-nets of a metric space).** Let (X, d) be a metric space. A subset $A \subseteq X$ is called an ϵ-net if $\{B_\epsilon(a) \mid a \in A\}$ is an open cover of X.

   Suppose that (X, d) is a sequentially compact metric space, and $\epsilon > 0$. Prove that X has a finite ϵ-net.

   (c) Let (X, d) be a sequentially compact metric space, and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. Show that there exists some finite collection $U_{i_1}, \ldots, U_{i_n} \in \mathcal{U}$ so that $\{U_{i_1}, \ldots, U_{i_n}\}$ covers X, i.e., so that $X = U_{i_1} \cup \cdots \cup U_{i_n}$.

   We will return to these results later in the course when we study *compactness*.