

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let (X, d) be a metric space. Verify that the collection \mathcal{T}_d of open sets in this metric space does indeed form a topology on the set X .
2. Find all possible topologies on the set $X = \{0, 1\}$.
3. Let $X = \{0, 1, 2\}$. Show that the collection of subsets $\{\emptyset, X, \{0, 1\}, \{1, 2\}\}$ is **not** a topology on X .
4. Let (X, \mathcal{T}) be a topological space.
 - (a) Show by induction that the intersection of any **finite** collection of open sets is open.
 - (b) Explain why this argument does not apply to an infinite collection of open sets.
5. Consider the following collection of subsets of \mathbb{R} . Note that \emptyset is considered a finite set.

$$\mathcal{R} = \{A \mid A \text{ is finite}\} \cup \{\mathbb{R}\}.$$

- (a) Show that $\emptyset, \mathbb{R} \in \mathcal{R}$.
 - (b) Show that if $U, V \in \mathcal{R}$, then so is $U \cap V$.
 - (c) Show that if $U, V \in \mathcal{R}$, then so is $U \cup V$. Show, in fact, that the union of any finite number of elements in \mathcal{R} is an element of \mathcal{R} .
 - (d) Show that arbitrary unions of elements in \mathcal{R} may not be in \mathcal{R} . Conclude that \mathcal{R} is not a topology on \mathbb{R} .
 - (e) Conclude that the second axiom of a topology—that it is closed under arbitrary unions—is truly stronger than the condition that it be closed under finite unions.
6. Let X be a set. Show that the discrete topology on X is induced by the discrete metric on X .
 7. Let X be a set. Show that if X is a finite set, then the cofinite topology (defined in Assignment Problem 3) coincides with the discrete topology on X .
 8. Let X be a topological space consisting of either 0 or 1 points. Explain why X satisfies both the T_1 and T_2 properties.

Worksheet problems

(Hand these questions in!)

- Worksheet #9 Problems 3, 4.

Assignment questions

(Hand these questions in!)

1. (a) **Definition (T_1 -space).** Let (X, \mathcal{T}) be a topological space. Then we say that X has the T_1 property, or call X a T_1 -space, if it satisfies the following condition: For every pair of distinct points $x, y \in X$, there exists some neighbourhood U_x of x that does not contain y , and there exists some neighbourhood U_y of y that does not contain x .

Prove that (X, \mathcal{T}) is a T_1 -space if and only if, for every point $x \in X$, the singleton set $\{x\}$ is closed. (Mathematicians often refer to this property by the slogan “points are closed”).

- (b) **Definition (Hausdorff property; T_2 -space).** Let (X, \mathcal{T}) be a topological space. Then we say that X has the T_2 property, or call X a Hausdorff space, if it satisfies the following condition: For every pair of distinct points $x, y \in X$, there exists open neighbourhoods U_x of x and U_y of y such that U_x and U_y are disjoint.

Show that every space satisfying the T_2 property must satisfy the T_1 property. Conclude in particular that, in a Hausdorff space, singleton sets $\{x\}$ are closed.

2. Let (X, \mathcal{T}) be a **finite** topological space. Show that, if X has the T_1 property, then the topology on X must be the discrete topology.
3. (a) Let X be a nonempty set, and let \mathcal{T} be the collection of subsets

$$\mathcal{T} = \{\emptyset\} \cup \{U \subseteq X \mid X \setminus U \text{ is a finite set}\}.$$

Verify that \mathcal{T} is a topology on X . It is called the *cofinite* topology (a contraction of “complements are finite”).

- (b) Consider the set \mathbb{R} with the cofinite topology. Verify that $(\mathbb{R}, \text{cofinite})$ is a T_1 -space but not Hausdorff. This shows that the T_2 condition is strictly stronger than the T_1 condition.
- (c) Conclude that the cofinite topology is not metrizable. *Hint:* Worksheet #4 Problem 2(a).