Final Exam

Math 490
16 December 2021
Sarah Koch & Jenny Wilson

Name: ____

Instructions: This exam has 10 questions for a total of 40 points.

Each student may bring in one double-sided ($8\frac{1}{2}$" $\times$ 11") sheet of notes, which they must have either hand-written or typed (in font size at least 12) themselves.

The exam is closed-book. No books, additional notes, cell phones, calculators, or other devices are permitted. Scratch paper is available.

Fully justify your answers unless otherwise instructed. You may cite any (non-optional) results proved on the worksheets, on a quiz, or on the homeworks without proof. Please include a complete statement of the result you are quoting.

You have 120 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Question	Points	Score
1	8	
2	1	
3	4	
4	4	
5	4	
6	6	
7	2	
8	1	
9	6	
10	4	
Total:	40	

1. (8 points) For each of the following statements: if the statement is always true, write "`True`". Otherwise, state a counterexample. **No further justification needed.**

   Note: If the statement is not always true, you can receive partial credit for writing "`False`" without a counterexample.

   (a) Let X be a metric space, $x, y \in X$, and $r > 0$. If $d(x, y) > 2r$, then the balls $B_r(x)$ and $B_r(y)$ are disjoint.

   (b) Let X and Y be metric spaces. If $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are Cauchy sequences in X and Y, respectively, then $((x_n, y_n))_{n\in\mathbb{N}}$ is Cauchy with respect to the product metric on $X \times Y$.

   (c) Let X and Y be topological spaces, and $\mathcal{B}$ a basis for the topology on X. Then a function $f : X \to Y$ is open if and only if $f(B)$ is open for every $B \in \mathcal{B}$.

   (d) Let A, B be **disjoint** subsets of a topological space X. Then $\partial(A \cup B) = \partial A \cup \partial B$.

   (e) Let X be a topological space with the property that every sequence converges (to at least one point). Then X must have the indiscrete topology.

   (f) Let X and Y be T_1-spaces. Then the product topology on $X \times Y$ has the T_1 property.

(g) Let $f : X \to Y$ be a continuous function of topological spaces. If X is metrizable, then $f(X)$ is metrizable.

(h) Let X be a complete metric space. Then any closed and bounded subset S of X is compact.

2. (1 point) Let $X = \{a, b, c, d\}$ with the topology $\mathcal{T} = \big\{\varnothing, \{c\}, \{c, b\}, \{a, c\}, \{a, b, c\}, \{a, b, c, d\}\big\}$. Write the subspace topology on the subspace $S = \{a, b, d\}$. **No justification needed.**

3. (4 points) Consider the following statement.

   Let $f : X \to Y$ be a continuous function of topological spaces.

   *If the space $f(X) \subseteq Y$ (with the subspace topology) is ____________,
   then so is X.*

   Circle all properties that truthfully fill in the blank. **No justification needed.**

   T_1 `Hausdorff` `connected` `disconnected`

   `indiscrete` `discrete` `compact` `noncompact`

   (By "X is discrete" we mean "X has the discrete topology". Similarly for indiscrete.)

4. (4 points) Consider the following topological spaces X and their subsets S. In each case, compute the interior $\mathrm{Int}(S)$, the closure $\overline{S}$, the boundary ∂S, and the set S' of accumulation points of S. **No justification necessary.**

   (a) Let $X = \{a, b, c, d\}$ with the topology $\Big\{\varnothing, \{a\}, \{b, c\}, \{a, b, c\}, \{a, d\}, \{a, b, c, d\}\Big\}$. Let $S = \{a, b\}$.

$\mathrm{Int}(S)$: __________ $\overline{S}$: __________ ∂S: __________ S': __________

   (b) Let $X = \mathbb{R}$ with the topology $\mathcal{T} = \{U \mid 0 \notin U\} \cup \{\mathbb{R}\}$. Let $S = \{0, 1\}$.

$\mathrm{Int}(S)$: __________ $\overline{S}$: __________ ∂S: __________ S': __________

5. (4 points) For each of the following sequences: state the set of all limits, or, if the sequence has no limits, write "`Does not converge`". **No justification necessary.**

   (a) Let $\mathbb{R}$ have the topology $\{A \mid 0 \in A\} \cup \{\varnothing\}$.

      (i) $0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ \cdots$

      (ii) $0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ \cdots$

   (b) Let $\mathbb{R}$ have the topology $\{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\mathbb{R}\} \cup \{\varnothing\}$

      (i) $0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ \cdots$

      (ii) $0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ \cdots$

6. (6 points) Circle all terms that apply. **No justification necessary.**

   (a) The subspace $\mathbb{Q} \subseteq \mathbb{R}$ with the standard topology is ...

   `compact` `connected` T_1 T_2 `(Hausdorff)`

   (b) The topology $\mathcal{T} = \{U \mid 0 \in U\} \cup \{\varnothing\}$ on $\mathbb{R}$ is ...

   `compact` `connected` T_1 T_2 `(Hausdorff)`

   (c) The set $X = \{a, b, c\}$ with the topology $\{\varnothing, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ is ...

   `compact` `connected` T_1 T_2 `(Hausdorff)`

7. (2 points) For each of the following maps f, circle all properties that apply.

   (a) $f : (\mathbb{R}, \text{ cofinite}) \to (\mathbb{R}, \text{ cofinite})$, $f(x) = |x|$ `continuous` `open`

   (b) Let $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\mathbb{R}\} \cup \{\varnothing\}$. $f : (\mathbb{R}, \mathcal{T}) \to (\mathbb{R}, \mathcal{T})$, $f(x) = |x|$ `continuous` `open`

8. (1 point) Let $X = \{a, b, c, d\}$ with the topology $\Big\{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}\Big\}$. Write down a path in X from a to b. **No justification necessary.**

9. (a) (3 points) Let X and Y be path-connected topological spaces. Show that the product $X \times Y$ (with the product topology) is path-connected.

(b) (3 points) Let X be a topological space, and let a and b be points in two distinct connected components of X. Show that there is no path from a to b.

10. (4 points) Suppose that X is a compact, Hausdorff topological space. Show that X satisfies the following property: For every point $x \in X$ and closed subset $C \subseteq X$ that does not contain x, there exist disjoint open subsets V and U of X such that $x \in V$ and $C \subseteq U$.

Blank page for extra work.