Final Exam

Math 490
16 December 2021
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Name: ______________________________

Instructions: This exam has 10 questions for a total of 40 points.

Each student may bring in one double-sided ($8\frac{1}{2}$" $\times$ 11") sheet of notes, which they must have either hand-written or typed (in font size at least 12) themselves.

The exam is closed-book. No books, additional notes, cell phones, calculators, or other devices are permitted. Scratch paper is available.

Fully justify your answers unless otherwise instructed. You may cite any (non-optional) results proved on the worksheets, on a quiz, or on the homeworks without proof. Please include a complete statement of the result you are quoting.

You have 120 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Question	Points	Score
1	8	
2	1	
3	4	
4	4	
5	4	
6	6	
7	2	
8	1	
9	6	
10	4	
Total:	40	

1. (8 points) For each of the following statements: if the statement is always true, write "`True`". Otherwise, state a counterexample. **No further justification needed.**

   Note: If the statement is not always true, you can receive partial credit for writing "`False`" without a counterexample.

   (a) Let X be a metric space, $x, y \in X$, and $r > 0$. If $d(x, y) > 2r$, then the balls $B_r(x)$ and $B_r(y)$ are disjoint.

   True. *Hint:* Suppose z were contained in both $B_r(x)$ and $B_r(y)$. Apply the triangle inequality to x, y, z.

   (b) Let X and Y be metric spaces. If $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are Cauchy sequences in X and Y, respectively, then $((x_n, y_n))_{n\in\mathbb{N}}$ is Cauchy with respect to the product metric on $X \times Y$.

   True. *Hint:* Proceed by direct calculation using the definition of the product metric. One approach is to use the observation,

$$\begin{aligned} d_{X\times Y}((x_m, y_m), (x_n, y_n)) &< d_{X\times Y}((x_m, y_m), (x_m, y_n)) + d_{X\times Y}((x_m, y_n), (x_n, y_n)) \\ &= d_Y(y_m, y_n) + d_X(x_m, x_n) \end{aligned}$$

   (c) Let X and Y be topological spaces, and $\mathcal{B}$ a basis for the topology on X. Then a function $f : X \to Y$ is open if and only if $f(B)$ is open for every $B \in \mathcal{B}$.

   True. *Hint:* Under these assumptions, $f\left(\bigcup_{i\in I} B_i\right) = \bigcup_{i\in I} f(B_i)$ is open for any union $\bigcup_{i\in I} B_i$ of basis elements.

   (d) Let A, B be disjoint subsets of a topological space X. Then $\partial(A \cup B) = \partial A \cup \partial B$.

   False. Let $X = \mathbb{R}$ with the standard topology, A be the rationals and B the irrationals. Then $\partial(A \cup B) = \partial\mathbb{R} = \varnothing$, but $\partial A \cup \partial B = \mathbb{R} \cup \mathbb{R} = \mathbb{R}$.

   (e) Let X be a topological space with the property that every sequence converges (to at least one point). Then X must have the indiscrete topology.

   False. For example, consider $X = \{0, 1\}$ with the topology $\mathcal{T} = \Big\{\varnothing, \{0\}, \{0, 1\}\Big\}$. This is not the indiscrete topology, but every sequence converges to 1.

   (f) Let X and Y be T_1-spaces. Then the product topology on $X \times Y$ has the T_1 property.

   True. *Hint:* Given distinct $(x_1, y_1), (x_2, y_2) \in X \times Y$, either $x_1 \neq x_2$ or $y_1 \neq y_2$. Suppose WLOG $x_1 \neq x_2$. Let U be an open neighbourhood of x_1 not containing x_2. Then $U \times Y$ is an open neighbourhood of (x_1, y_1) not containing (x_2, y_2).

(g) Let $f : X \to Y$ be a continuous function of topological spaces. If X is metrizable, then $f(X)$ is metrizable.

False. Consider, for example, the identity function from $X = \mathbb{R}$ with the Euclidean topology to $Y = \mathbb{R}$ with the indiscrete topology. The function is continuous by Worksheet #11 Problem 2(b), and the topology on X is induced by the Euclidean metric. In contrast $f(X) = Y$ is not metrizable by Worksheet #9 Problem 2(b).

(h) Let X be a complete metric space. Then any closed and bounded subset S of X is compact.

False. For example, let $X = S = \mathbb{R}$ with the discrete metric. Then X is complete and S is closed and bounded in X, but S is noncompact by Worksheet #17 Example 1.5.

2. (1 point) Let $X = \{a, b, c, d\}$ with the topology $\mathcal{T} = \big\{\varnothing, \{c\}, \{c, b\}, \{a, c\}, \{a, b, c\}, \{a, b, c, d\}\big\}$. Write the subspace topology on the subspace $S = \{a, b, d\}$. **No justification needed.**

Solution. By definition, the subspace topology $\mathcal{T}_S$ is the collection of all intersections of S with each element of $\mathcal{T}$. We find

$$\mathcal{T}_S = \big\{\varnothing, \{b\}, \{a\}, \{a, b\}, \{a, b, d\}\big\}.$$

3. (4 points) Consider the following statement.

Let $f : X \to Y$ be a continuous function of topological spaces.

If the space $f(X) \subseteq Y$ (with the subspace topology) is ____________, then so is X.

Circle all properties that truthfully fill in the blank. **No justification needed.**

T_1	`Hausdorff`	`connected`	`disconnected`
`indiscrete`	`discrete`	`compact`	`noncompact`

(By "X is discrete" we mean "X has the discrete topology". Similarly for indiscrete.)

Hint: Since a single-point subspace $\{y\}$ is necessarily T_1, Hausdorff, indiscrete, discrete, connected, and compact, you can construct counterexamples to all of these options using a constant map from a suitably chosen space X.

4. (4 points) Consider the following topological spaces X and their subsets S. In each case, compute the interior $\mathrm{Int}(S)$, the closure $\overline{S}$, the boundary ∂S, and the set S' of accumulation points of S. **No justification necessary.**

   (a) Let $X = \{a, b, c, d\}$ with the topology $\Big\{\varnothing, \{a\}, \{b, c\}, \{a, b, c\}, \{a, d\}, \{a, b, c, d\}\Big\}$. Let $S = \{a, b\}$.

   $\mathrm{Int}(S)$: $\{a\}$ $\overline{S}$: $\{a, b, c, d\}$ ∂S: $\{b, c, d\}$ S': $\{c, d\}$

   (b) Let $X = \mathbb{R}$ with the topology $\mathcal{T} = \{U \mid 0 \notin U\} \cup \{\mathbb{R}\}$. Let $S = \{0, 1\}$.

   $\mathrm{Int}(S)$: $\{1\}$ $\overline{S}$: $\{0,1\}$ ∂S: $\{0\}$ S': $\{0\}$

5. (4 points) For each of the following sequences: state the set of all limits, or, if the sequence has no limits, write "`Does not converge`". **No justification necessary.**

   (a) Let $\mathbb{R}$ have the topology $\{A \mid 0 \in A\} \cup \{\varnothing\}$.

   (i) $0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ \cdots$ limits: $\{1\}$

   (ii) $0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ \cdots$ Does not converge.

   (b) Let $\mathbb{R}$ have the topology $\{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\mathbb{R}\} \cup \{\varnothing\}$

   (i) $0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ \cdots$ limits: $(-\infty, 0]$

   (ii) $0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ \cdots$ limits: all real numbers

6. (6 points) Circle all terms that apply. **No justification necessary.**

   (a) The subspace $\mathbb{Q} \subseteq \mathbb{R}$ with the standard topology is ...

   `compact` `connected` T_1 T_2 `(Hausdorff)`

   (b) The topology $\mathcal{T} = \{U \mid 0 \in U\} \cup \{\varnothing\}$ on $\mathbb{R}$ is ...

   `compact` `connected` T_1 T_2 `(Hausdorff)`

   (c) The set $X = \{a, b, c\}$ with the topology $\{\varnothing, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ is ...

   `compact` `connected` T_1 T_2 `(Hausdorff)`

7. (2 points) For each of the following maps f, circle all properties that apply.

   (a) $f : (\mathbb{R},\ \text{cofinite}) \to (\mathbb{R},\ \text{cofinite})$
   $f(x) = |x|$ `continuous` `open`

   (b) Let $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\mathbb{R}\} \cup \{\varnothing\}$.
   $f : (\mathbb{R}, \mathcal{T}) \to (\mathbb{R}, \mathcal{T})$
   $f(x) = |x|$ `continuous` `open`

8. (1 point) Let $X = \{a, b, c, d\}$ with the topology $\Big\{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}\Big\}$. Write down a path in X from a to b. **No justification necessary.**

   Solution. One possible solution is

$$f(x) = \begin{cases} a, & x \in [0, \frac{1}{2}) \\ c, & x = \frac{1}{2} \\ b, & x \in (\frac{1}{2}, 1] \end{cases}$$

   To verify it is continuous, we check that the preimage of every open subset of X is open in $[0, 1]$. (This work does not need to be shown in a student's solution).

$$f^{-1}(\{a\}) = \left[0, \frac{1}{2}\right) \quad f^{-1}(\{b\}) = \left(\frac{1}{2}, 1\right] \quad f^{-1}(\{a, b\}) = \left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right]$$
$$f^{-1}(\{a, b, c\}) = [0, 1] \quad f^{-1}(\{a, b, c, d\}) = [0, 1]$$

9. (a) (3 points) Let X and Y be path-connected topological spaces. Show that the product $X \times Y$ (with the product topology) is path-connected.

Solution. Let (x_1, y_1) and (x_2, y_2) be any two points in $X \times Y$. Since X is path-connected by assumption, there exists a path $\gamma_X(t)$ in X from x_1 to x_2. Similarly there exists a path $\gamma_Y(t)$ in Y from y_1 to y_2.
Consider the function

$$\begin{aligned} \gamma : [0,1] &\to X \times Y \\ \gamma(t) &= (\gamma_X(t), \gamma_Y(t)) \end{aligned}$$

Since γ_X and γ_Y are continuous, by Worksheet #13 Problem 3, the map γ is continuous. Moreover, $\gamma(0) = (\gamma_X(0), \gamma_Y(0)) = (x_1, y_1)$ and $\gamma(1) = (\gamma_X(1), \gamma_Y(1)) = (x_2, y_2)$. Thus γ is a path from (x_1, y_1) to (x_2, y_2). Since these points were arbitrary, we conclude that $X \times Y$ is path-connected.

(b) (3 points) Let X be a topological space, and let a and b be points in two distinct connected components of X. Show that there is no path from a to b.

Solution. Let A and B be the connected components containing a and b, respectively. We proved on Homework #12 Problem 4(c) that they are disjoint.
Suppose that $\gamma : [0,1] \to X$ were a path from a to b. By Homework #12 Problem 2(b), its domain $[0,1]$ is connected, thus by Homework #12 Problem 2(a) its image $\gamma([0,1])$ is connected.
The intersection $A \cap \gamma([0,1])$ contains a and is therefore nonempty. We proved on Worksheet #15 Problem 6(b) that the union of two connected subsets with nonempty intersection is connected. We deduce that $A \cup \gamma([0,1])$ is connected. But $A \subseteq A \cup \gamma([0,1])$, so by definition of a connected component, $A = A \cup \gamma([0,1])$. This is a contradiction, since $b \in \gamma([0,1])$ but $b \notin A$.

10. (4 points) Suppose that X is a compact, Hausdorff topological space. Show that X satisfies the following property: For every point $x \in X$ and closed subset $C \subseteq X$ that does not contain x, there exist disjoint open subsets V and U of X such that $x \in V$ and $C \subseteq U$.

Solution. A topological space satisfying this property is called *regular*.

Suppose that X is compact and Hausdorff. We will prove that X is regular using a strategy similar to the solution to Worksheet #17 Problem 4(b).

Fix a point x in X and a disjoint closed subset $C \subseteq X$. Since X is Hausdorff, for each point c in C we can find disjoint open subsets U_c and V_c such that $c \in U_c$ and $x \in V_c$. The subsets $\{U_c \mid c \in C\}$ are an open cover of C, since for any point $c \in C$ we have $c \in U_c$ by construction.

The space X is compact by assumption, and we proved on Worksheet #17 Problem 3 that closed subsets of compact spaces are compact. Since C is compact, by definition, there must exist a finite collection of points $c_1, \ldots, c_n$ such that the subcover $\{U_{c_1}, \ldots, U_{c_n}\}$ covers C.

We claim that

$$U = U_{c_1} \cup U_{c_2} \cup \cdots \cup U_{c_n} \qquad \text{and} \qquad V = V_{c_1} \cap V_{c_2} \cap \cdots \cap V_{c_n}$$

are the desired open subsets of X. The statement that $\{U_{c_1}, \ldots, U_{c_n}\}$ covers C means that $C \subseteq U$. And, since $x \in V_c$ for all c, we know x is contained in the intersection $V = V_{c_1} \cap V_{c_2} \cap \cdots \cap V_{c_n}$. We know that U and V are open because they are the union and intersection, respectively, of finitely many open subsets.

It remains to check that U and V are disjoint. It suffices to show that an arbitrary point $u \in U$ is not contained in V. If $u \in U$, then u must be contained in U_{c_i} for some i. But U_{c_i} and V_{c_i} are disjoint by construction. So $u \notin V_{c_i}$ and therefore u is not contained in its subset $V \subseteq V_{c_i}$.

Thus U and V are disjoint open subsets of X satisfying $C \subseteq U$ and $x \in V$ as claimed.

Blank page for extra work.