Midterm Exam

Math 490
26 October 2021
Sarah Koch & Jenny Wilson

Name: ______________________________

Instructions: This exam has 5 questions for a total of 20 points.

The exam is closed-book. No books, cell phones, calculators, or other devices are permitted.

Each student may bring in one double-sided standard-size (8.5 in $\times$ 11 in) sheet of notes, which they must prepare themselves. Notes may be handwritten or typed with font size at least 12.

Scratch paper is available.

Fully justify your answers unless otherwise instructed. You may quote any results proved in class, on a quiz, or on the homeworks without proof. Please include a complete statement of the result you are quoting.

You have 80 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Sarah or Jenny is available to answer questions.

Question	Points	Score
1	6	
2	4	
3	3	
4	3	
5	4	
Total:	20	

1. (6 points) For each of the following statements: if the statement is always true, write "`True`". Otherwise, state a counterexample. **No further justification needed.**

   Note: If the statement is not always true, you can receive partial credit for writing "`False`" without a counterexample.

   (a) Suppose that X and Y are homeomorphic metric spaces. If X is bounded, then Y is bounded.

   (b) Suppose that X and Y are homeomorphic metric spaces. If X is sequentially compact, then Y is sequentially compact.

   (c) Let X be a metric space, and $A, B \subseteq X$ be subsets. If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.

   (d) Let X be a metric space, and $A, B \subseteq X$ be subsets. If $A \subseteq B$, then $\partial A \subseteq \partial B$.

(e) Let S be a sequentially compact subset of a metric space X. Then S contains the limits of every convergent sequence of points in S.

(f) Let C be a closed, bounded, nonempty subset of a metric space X. Then any continuous function $f : X \to \mathbb{R}$ achieves its maximum on C.

2. (4 points) Below are two metric spaces X and subsets A. For each subset, state the interior, closure, and boundary of A, and its set A' of accumulation points. **No justification needed.**

$X = \mathbb{R}$ with the Euclidean metric, A is the subset of **irrational** real numbers.

$\text{Int}(A) =$ __________ $\overline{A} =$ __________ $\partial A =$ __________ $A' =$ __________

$X = \mathbb{R}$ with the Euclidean metric, A any nonempty **finite** subset.

$\text{Int}(A) =$ __________ $\overline{A} =$ __________ $\partial A =$ __________ $A' =$ __________

3. (3 points) Let (X, d) be a metric space, and let $A \subseteq X$ be subset. Prove that its interior $\operatorname{Int}(A)$ is equal to the union of all subsets of A that are open in X.

4. (3 points) Let S be a subset of $\mathbb{R}$, viewed as a metric space under the restriction of the Euclidean metric. Show that S is complete if and only if it is a closed subset of $\mathbb{R}$.

5. (4 points) Let (X, d_X), (Y, d_Y), and (Z, d_Z) be nonempty metric spaces, and endow the product $X \times Y$ with the product metric $d_{X \times Y}$. Prove that a function

$$f : Z \longrightarrow X \times Y$$

is continuous if and only if it satisfies the following condition:

The preimage $f^{-1}(U \times V)$ is open for every subset of $X \times Y$ of the form $U \times V$ with $U \subseteq X$ an open subset of X and $V \subseteq Y$ an open subset of Y.

Blank page for extra work