Midterm Exam

Math 490
26 October 2021
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Name: ______________________________

Instructions: This exam has 5 questions for a total of 20 points.

The exam is closed-book. No books, cell phones, calculators, or other devices are permitted.

Each student may bring in one double-sided standard-size (8.5 in $\times$ 11 in) sheet of notes, which they must prepare themselves. Notes may be handwritten or typed with font size at least 12.

Scratch paper is available.

Fully justify your answers unless otherwise instructed. You may quote any results proved in class, on a quiz, or on the homeworks without proof. Please include a complete statement of the result you are quoting.

You have 80 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Sarah or Jenny is available to answer questions.

Question	Points	Score
1	6	
2	4	
3	3	
4	3	
5	4	
Total:	20	

1. (6 points) For each of the following statements: if the statement is always true, write "`True`". Otherwise, state a counterexample. **No further justification needed.**

   Note: If the statement is not always true, you can receive partial credit for writing "`False`" without a counterexample.

   (a) Suppose that X and Y are homeomorphic metric spaces. If X is bounded, then Y is bounded.

   False. For example, the functions

$$\begin{aligned}\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) &\longrightarrow \mathbb{R} \\ x &\longmapsto \tan(x) \\ \arctan(x) &\longleftarrow\!\shortmid\; x\end{aligned}$$

   are homeomorphisms between the spaces $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\mathbb{R}$ with the Euclidean metrics. However, the first is bounded and the second is not.

   Alternate example: consider $f(x) = \frac{1}{x}$ from $(0,1)$ to $(1,\infty)$.

   (b) Suppose that X and Y are homeomorphic metric spaces. If X is sequentially compact, then Y is sequentially compact.

   True. *Hint:* Let $f : X \to Y$ be a homeomorphism. By Homework #6 Problem 1(d), $Y = f(X)$ is sequentially compact.

   (c) Let X be a metric space, and $A, B \subseteq X$ be subsets. If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.

   True. *Hint:* See Worksheet #5 Theorem 1.8 (*iii*).

   (d) Let X be a metric space, and $A, B \subseteq X$ be subsets. If $A \subseteq B$, then $\partial A \subseteq \partial B$.

   False. For example, let $X = \mathbb{R}$ with the standard topology, and let $A = \mathbb{Q}$, $B = \mathbb{R}$. Then $A \subseteq B$, but $\partial A = \mathbb{R}$ and $\partial B = \varnothing$.

(e) Let S be a sequentially compact subset of a metric space X. Then S contains the limits of every convergent sequence of points in S.

True. *Hint:* Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of points in S converging to some $x \in X$. Since S is sequentially compact, some subsequence converges to a point $s \in S$. But $s = x$ by Worksheet 6 Proposition 1.2.

Alternatively, S is closed by Worksheet 6 Problem 2(a), so the result follows from Homework #3 Problem 3.

(f) Let C be a closed, bounded, nonempty subset of a metric space X. Then any continuous function $f : X \to \mathbb{R}$ achieves its maximum on C.

False. We proved this holds for nonempty sequentially compact subsets C, but sequential compactness is not equivalent to closed/boundedness in general. For example, let $C = X$ be the natural numbers $\mathbb{N}$ with the discrete metric, and consider the continuous function

$$\begin{aligned} f : X &\longrightarrow \mathbb{R} \\ n &\longmapsto n \end{aligned}$$

Then C is nonempty, closed, and bounded, but f does not have a finite maximum on C. Alternate example: consider the inclusion of $C = X = (0, 1)$ into $\mathbb{R}$.

2. (4 points) Below are two metric spaces X and subsets A. For each subset, state the interior, closure, and boundary of A, and its set A' of accumulation points. **No justification needed.**

$X = \mathbb{R}$ with the Euclidean metric, A is the subset of **irrational** real numbers.

$\text{Int}(A) = \varnothing$ $\quad\overline{A} = \mathbb{R}$ $\quad\partial A = \mathbb{R}$ $\quad A' = \mathbb{R}$

$X = \mathbb{R}$ with the Euclidean metric, A any nonempty **finite** subset.

$\text{Int}(A) = \varnothing$ $\quad\overline{A} = A$ $\quad\partial A = A$ $\quad A' = \varnothing$

3. (3 points) Let (X, d) be a metric space, and let $A \subseteq X$ be subset. Prove that its interior $\mathrm{Int}(A)$ is equal to the union of all subsets of A that are open in X.

Solution. Compare to Quiz #5 Problem 1(b).

Let $U = \bigcup_{\substack{V \subseteq A \\ V \text{ open in } X}} V$. Our goal is to show that $U = \mathrm{Int}(A)$.

By Worksheet #5 Theorem 1.7 (i), $\mathrm{Int}(A) \subseteq A$. By Worksheet #5 Theorem 1.7 (v), $\mathrm{Int}(A)$ is open in X. Thus $V = \mathrm{Int}(A)$ is an instance of an open subset contained in A. It follows that $\mathrm{Int}(A)$ must be contained in the union of all such subsets. We conclude that $\mathrm{Int}(A) \subseteq U$.

To show the opposite inclusion, let V be an open subset of X contained in A. But then, by Worksheet #5 Theorem 1.7 (vi), $V \subseteq \mathrm{Int}(A)$. If all such subsets V are contained in $\mathrm{Int}(A)$, it follows that their union U is contained in $\mathrm{Int}(A)$. So $U \subseteq \mathrm{Int}(A)$.

We have deduced that $\mathrm{Int}(A) = \bigcup_{\substack{V \subseteq A \\ V \text{ open in } X}} V$ as desired.

4. (3 points) Let S be a subset of $\mathbb{R}$, viewed as a metric space under the restriction of the Euclidean metric. Show that S is complete if and only if it is a closed subset of $\mathbb{R}$.

Solution. First suppose that S is not closed. We proved in Homework #3 Problem 3 that there must therefore be a sequence $(s_n)_{n \in \mathbb{N}}$ of points in S that converges to a point x with $x \notin S$. Since this sequence converges in $\mathbb{R}$, it is Cauchy by Homework #3 Problem 5(a). However, since limits of sequences in $\mathbb{R}$ are unique (Worksheet #4, Problem 2), the sequence does not converge to any point of S. Thus S is not complete.

Now suppose that S is closed, and suppose $(s_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in S. (If S is empty, there is nothing to check.) Since $\mathbb{R}$ is complete, the sequence converges to a point $x \in \mathbb{R}$. Since S is closed, by Homework #3 Problem 3, it must contain the limits of all of its convergent sequences. Thus $x \in S$. The sequence converges in S, and we conclude that S is complete as claimed.

5. (4 points) Let (X, d_X), (Y, d_Y), and (Z, d_Z) be nonempty metric spaces, and endow the product $X \times Y$ with the product metric $d_{X\times Y}$. Prove that a function

$$f : Z \longrightarrow X \times Y$$

is continuous if and only if it satisfies the following condition:

The preimage $f^{-1}(U \times V)$ is open for every subset of $X \times Y$ of the form $U \times V$ with $U \subseteq X$ an open subset of X and $V \subseteq Y$ an open subset of Y.

Solution. First suppose that f is continuous. By Worksheet #3 Problem 1, $f^{-1}(W)$ is open for every open subset $W \subseteq X \times Y$. By Worksheet #7 Problem 2(a), the subset $W = U \times V \subseteq X \times Y$ is open whenever $U \subseteq X$ and $V \subseteq Y$ are open. Thus $f^{-1}(U \times V)$ is open for all such subsets of $X \times Y$.

Now suppose that $f^{-1}(U \times V)$ is open for every subset of $X \times Y$ of the form $U \times V$ with $U \subseteq X$ and $V \subseteq Y$ both open. Our goal is to show that f is continuous. By Worksheet #3 Problem 1, it suffices to show that $f^{-1}(W)$ is an open subset of Z for all open $W \subseteq X \times Y$.

Let $W \subseteq X \times Y$ be an open subset. To show $f^{-1}(W)$ is open, we will show that an arbitrary point $z \in f^{-1}(W)$ is an interior point of $f^{-1}(W)$. Since $z \in f^{-1}(W)$, by definition, $f(z) \in W$. Since W is open, by Worksheet #7 Problem 2(b), there exists open sets $U \subseteq X$ and $V \subseteq Y$ such that $f(z) \in U \times V \subseteq W$.

It follows that $z \in f^{-1}(U \times V) \subseteq f^{-1}(W)$. By hypothesis, $f^{-1}(U \times V)$ is open. Then by Worksheet #5 Theorem 1.5, z is an interior point of $f^{-1}(W)$. We conclude that $f^{-1}(W)$ is open, and therefore that f is continuous, as claimed.