Name: ______________________________ Score (Out of 5 points):

1. (5 points) Let (X, d_X) and (Y, d_Y) be two metric spaces. Let $X \times Y$ be the Cartesian product of the sets X and Y, i.e., $X \times Y$ is the set

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}.$$

Prove that the following function defines a metric on $X \times Y$.

$$\begin{aligned} d : (X \times Y) \times (X \times Y) &\longrightarrow \mathbb{R} \\ d((x_1, y_1), (x_2, y_2)) &= d_X(x_1, x_2) + d_Y(y_1, y_2) \end{aligned}$$

Solutions. From the description of d we see that it is well-defined: its outputs are (uniquely) defined for every element of $(X \times Y) \times (X \times Y)$, and its outputs are always elements of $\mathbb{R}$. We do not need to do any additional work to verify well-definedness of d.

To show that d is a metric, we will check the 3 conditions that define a metric:

(M1) **(Positivity).** $d((x_1, y_1), (x_2, y_2)) \geq 0$ for all $(x_1, y_1), (x_2, y_2) \in X \times Y$, and $d((x_1, y_1), (x_2, y_2)) = 0$ if and only if $(x_1, y_1) = (x_2, y_2)$.

(M2) **(Symmetry).** $d((x_1, y_1), (x_2, y_2)) = d((x_2, y_2), (x_1, y_1))$ for all $(x_1, y_1), (x_2, y_2) \in X \times Y$.

(M3) **(Triangle inequality).** $d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)) \geq d((x_1, y_1), (x_3, y_3))$ for all $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$.

We will use the assumption that d_X, d_Y are metrics. We first check (M1). Let $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ be any points.

$$\begin{aligned} d((x_1, y_1), (x_2, y_2)) &= d_X(x_1, x_2) + d_Y(y_1, y_2) \\ &\geq 0 + 0 && \text{[since } d_X, d_Y \text{ satisfy (M1)]} \\ &= 0 \end{aligned}$$

$$\begin{aligned} & d((x_1, y_1), (x_2, y_2)) = 0 \\ \iff & d_X(x_1, x_2) + d_Y(y_1, y_2) = 0 \\ \iff & d_X(x_1, x_2) = 0 \text{ and } d_Y(y_1, y_2) = 0 && \text{[since both terms are nonnegative]} \\ \iff & x_1 = x_2 \text{ and } y_1 = y_2 && \text{[since } d_X, d_Y \text{ satisfy (M1)]} \\ \iff & (x_1, y_1) = (x_2, y_2) \end{aligned}$$

We next check (M2). Let $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ be any points.

$$\begin{aligned} d((x_1, y_1), (x_2, y_2)) &= d_X(x_1, x_2) + d_Y(y_1, y_2) \\ &= d_X(x_2, x_1) + d_Y(y_2, y_1) && \text{[since } d_X, d_Y \text{ satisfy (M2)]} \\ &= d((x_2, y_2), (x_1, y_1)) \end{aligned}$$

Finally, we check (M3). Let $x_1, x_2, x_3 \in X$ and $y_1, y_2, y_3 \in Y$ be any points.

$$\begin{aligned}
d((x_1, y_1),&(x_2, y_2)) + d((x_2, y_2), (x_3, y_3))) \\
&= d_X(x_1, x_2) + d_Y(y_1, y_2) + d_X(x_2, x_3) + d_Y(y_2, y_3) \\
&= \Big(d_X(x_1, x_2) + d_X(x_2, x_3)\Big) + \Big(d_Y(y_1, y_2) + d_Y(y_2, y_3)\Big) \\
&\geq d_X(x_1, x_3) + d_Y(y_1, y_3) && \text{[since } d_X, d_Y \text{ satisfy (M3)]} \\
&= d((x_1, y_1), (x_3, y_3))
\end{aligned}$$