Name: ______________________ Score (Out of 7 points):

1. Let X be a nonempty finite set, say, $X = \{x_1, x_2, \ldots, x_n\}$ for some $n \in \mathbb{N}$. Let d be a metric on X.

   (a) (4 points) Prove that, for any point $x_i \in X$, the singleton set $U = \{x_i\}$ is open.

   Solution. First suppose that X has a single point x_1. Then $U = \{x_1\} = X$ is open by Worksheet #2 Problem 1(a).

   Now suppose $n \geq 2$, and let $U = \{x_i\}$ for some x_i in X. To show that U is open, by definition, we must show that every one of its points is an interior point. This means, for each point in U, we must find an open ball centered at that point and contained in U.

   Since U only contains the single point x_i, our goal becomes the following: We wish to find some radius $r > 0$ so that the ball $B_r(x_i) \subseteq U = \{x_i\}$, in other words, so that the open ball $B_r(x_i)$ does not contain any point other than x_i.

   Let
   $$r = \min_{x_j \in X, j \neq i} d(x_i, x_j).$$
   Since r is the minimum of a (nonempty) finite list of strictly positive numbers, the minimum must exist, and it must be a strictly positive number. (This is where the assumption that X is finite is needed!)

   So consider the ball $B_r(x_i)$. For any point x_j distinct from x_i, $d(x_i, x_j) \geq r$ by choice of r, so $x_j \notin B_r(x_i) = \{x \in X \mid d(x, x_i) < r\}$.

   Thus $B_r(x_i) = \{x_i\} \subseteq U$, and we conclude that U is open.

   (b) (2 points) Prove that *every* subset $U \subseteq X$ is open.
   Hint: Write U as a union of singleton sets.

   Solution. Let $U \subseteq X$ be any subset. If U is empty, then U is open by Worksheet 2 Problem 1. If U is nonempty, then we can write[1]
   $$U = \bigcup_{x \in U} \{x\}.$$
   By part (a), each set $\{x\}$ is open. By Worksheet 2 Problem 4, a union of open sets is open. Thus U is open.

   (c) (1 point) Prove that every subset $U \subseteq X$ is closed.

   Solution. Let $U \subseteq X$ be any subset. We proved that every subset of X is open. In particular, the complement $X \setminus U$ of U is open. Thus, by definition, U is closed.

[1] By convention, the empty set is the union of an empty collection of sets, so this decomposition actually also holds when U is empty.