Name: ______________________ Score (Out of 4 points):

1. On this quiz we will prove the following theorem.

   Theorem. Let $f : X \to Y$ be a function of metric spaces. Then f is continuous if and only if the preimage of every open ball in Y is open. In other words, f is continuous if and only if $f^{-1}\Big(B_r(y)\Big)$ is an open subset of X for every $y \in Y$ and every $r > 0$.

   Let $f : X \to Y$ be a function between metric spaces.

   (a) (1 point) Suppose that f is continuous. Explain why the preimage $f^{-1}\Big(B_r(y)\Big)$ of a ball is an open subset of X for every $y \in Y$ and every $r > 0$.

   Solution. We proved on Worksheet 3 that the function f is continuous if and only if the preimage of every open set is open. Since an open ball $B_r(y)$ is open in Y, its preimage $f^{-1}\Big(B_r(y)\Big)$ under f is open.

   (b) (3 points) Suppose the function f has the property that $f^{-1}\Big(B_r(y)\Big)$ is an open subset of X for every y in Y and $r > 0$. Prove that f is continuous.

   Solution. Suppose that f has the property that $f^{-1}\Big(B_r(y)\Big)$ is an open subset of X for every y in Y and $r > 0$. Choose an arbitrary point $x \in X$. We will show that f is continuous at x.
   Fix $\epsilon > 0$, and consider the point $f(x) \in Y$. By assumption, the preimage $f^{-1}\Big(B_\epsilon(f(x))\Big)$ of the ball $B_\epsilon(f(x))$ is open.
   Because the ball $B_\epsilon(f(x))$ contains the point $f(x)$, by definition of preimage, its preimage $f^{-1}\Big(B_\epsilon(f(x))\Big)$ must contain the point $x \in X$. Because the preimage $f^{-1}\Big(B_\epsilon(f(x))\Big)$ is open, the point x must be an interior point. Hence, there exists some $\delta > 0$ so that

   $$B_\delta(x) \subseteq f^{-1}\Big(B_\epsilon(f(x))\Big).$$

   Let $x' \in B_\delta(x)$. Then $x' \in f^{-1}\Big(B_\epsilon(f(x))\Big)$. But this means (by definition of preimage) that $f(x') \in B_\epsilon(f(x))$. Hence the containment $B_\delta(x) \subseteq f^{-1}\Big(B_\epsilon(f(x))\Big)$ implies that

   $$f\Big(B_\delta(x)\Big) \subseteq B_\epsilon(f(x)).$$

   We have therefore proved that, for all $x \in X$ and all $\epsilon > 0$, there exists a $\delta > 0$ so that $f\Big(B_\delta(x)\Big) \subseteq B_\epsilon(f(x))$. This means, by definition, that the function f is continuous, as claimed.