Name: ____________________ Score (Out of 6 points):

1. (6 points) Let (X, d) be a metric space, and let $x, y \in X$ be two distinct points. Consider the alternating sequence

$$\begin{aligned} a_1 &= x \\ a_2 &= y \\ a_3 &= x \\ a_4 &= y \\ a_5 &= x \\ a_6 &= y \\ &\vdots \end{aligned}$$

Prove that the sequence $(a_n)_{n\in\mathbb{N}}$ does **not** converge.

Solution: We proved on Worksheet #4 Problem 2 that metric spaces have the *Hausdorff* property: given an two distinct points a and b, there exist disjoint open subsets U_a and U_b with U_a a neighbourhood of a and U_b a neighbourhood of b. In particular we can always find a neighbourhood of a that does not contain b.

To show that this sequence does not converge, we must show that, for every point $z \in X$, the sequence does not converge to z. We will consider three cases: the case that $z = x$, the case that $z = y$, and the case that z is neither x nor y.

We proved in Worksheet #4 Problem 1 that a sequence $(a_n)_{n\in\mathbb{N}}$ converges to a point $z \in X$ if, for every neighbourhood U of z, there exists some $N \in \mathbb{N}$ such that $a_n \in U$ for every $n \geq N$. Thus, to show that $(a_n)_{n\in\mathbb{N}}$ does not converge to a point z, we must find a neighbourhood U of z such that for every N there is some $n \geq N$ with $a_n \notin U$. In other words, there are terms a_n in the sequence with arbitrarily large index n that are not contained in U.

We first show that the sequence does not converge to x. By the Hausdorff property of X, we can find an open subset U of x which does not contain y. But then a_n is not contained in U for infinitely many values of n. Specifically, for any $N \in \mathbb{N}$, we have $2N > N$ and $a_{2N} = y \notin U$.

By reversing the roles of x and y, this same argument shows that the sequence does not converge to y.

Now suppose z is a point distinct from both x and y. By the Hausdorff property, we can choose a neighbourhood U of z that does not contain x. Then a_n is not contained in U for infinitely many values of n, and we conclude that $(a_n)_{n\in\mathbb{N}}$ does not converge to z.

Thus we have shown, for each $z \in X$, that the sequence $(a_n)_{n\in\mathbb{N}}$ does not converge to z. We conclude that the sequence does not converge, as claimed.

Alternate Solution Outline: We can show the sequence fails to be Cauchy, by considering $\epsilon = d(x, y) > 0$. By Homework #3 Asssignment Problem 5, a sequence that is not Cauchy cannot converge.