Name: ____________________________ Score (Out of **??** points):

1. Let (X, d) be a metric space. On this quiz we will prove two equivalent characterizations of the closure $\overline{A}$ of a subset $A \subseteq X$.

   (a) (4 points) Let $A \subseteq X$. Let A' denote the set of all accumulation points of A. Prove that

$$\overline{A} = A \cup A'.$$

Solution. Recall from the homework that an *accumulation point* x of A is, by definition, a point with the property that every open ball $B_r(x)$ centred on x contains a point of A distinct from x. Recall moreover, by definition, that a point x is in the *closure* of A iff every open ball $B_r(x)$ centred on x contains a point of A.

We will first show that $A \cup A' \subseteq \overline{A}$. We proved on Worksheet # 5 Theorem 1.8 (i) that $A \subseteq \overline{A}$. Hence it suffices to show that $A' \subseteq \overline{A}$. Let x be an accumulation point of A. By definition every neighbourhood of x contains a point of A (which happens to be distinct from x), so by definition $x \in \overline{A}$.

Next we will check that $\overline{A} \subseteq A \cup A'$. Let $x \in \overline{A}$. If $x \in A$ then evidently $x \in A \cup A'$. So suppose that x is not in A. Fix $r > 0$. By definition of closure, the ball $B_r(x)$ must contain a point $a \in A$. Since x itself is not in a, we must have $a \neq x$. Thus for all $r > 0$ the ball $B_r(x)$ contains a point of A distinct from x, and we conclude that x is an accumulation point of A. Thus $x \in A \cup A'$.

   (b) (4 points) Let $A \subseteq X$. Prove that $\overline{A}$ is equal to the intersection of all closed subsets containing A.

Solution. Let $I = \bigcap_{\substack{C \text{ closed} \\ A \subseteq C}} C$. Our goal is to show $I = \overline{A}$.

We first show $I \subseteq \overline{A}$. Let $x \in I$. We proved on Worksheet 5, Theorem 1.8 (i) that $A \subseteq \overline{A}$, and in Theorem 1.8 (v) that $\overline{A}$ is closed. Thus, $C = \overline{A}$ is an instance of a closed set containing A. Since x is contained in the intersection of all such sets, it must be contained in $\overline{A}$ in particular.

We now show that $\overline{A} \subseteq I$. We proved on Worksheet 5, Theorem 1.8 (vi) that if C is a closed subset containing A, then $\overline{A} \subseteq C$. Since $\overline{A}$ is contained in every such subset C, it is contained in their intersection. Thus $\overline{A} \subseteq I$ as claimed.