Name: ______________________ Score (Out of 5 points):

1. (3 points) Let (X, d) be a metric space, and let $A, B \subseteq X$. Show that, if A and B are sequentially compact, then so is $A \cap B$.

Solution 1. Suppose that A is sequentially compact. This means that every sequence in A has a subsequence that converges to a point of A. Assume too that B is sequentially compact.

Our goal is to show that $A \cap B$ is sequentially compact. So let $(a_n)_{n \in \mathbb{N}}$ be a sequence of points in $A \cap B$. Since A is sequentially compact, by definition, there is some subsequence $(a_{n_i})_{i \in \mathbb{N}}$ converging to a point $a \in A$. But, since B is sequentially compact, we proved on Worksheet #6 Problem 2 that B is closed. Since $(a_{n_i})_{i \in \mathbb{N}}$ is a convergent sequence in the closed set B, we proved moreover on Homework #3 Problem 3 that its limit $a = \lim_{i \in \mathbb{N}} a_{n_i}$ must be in B.

So $a \in A \cap B$. Thus we have found a subsequence $(a_{n_i})_{i \in \mathbb{N}}$ of $(a_n)_{n \in \mathbb{N}}$ converging to a point of $A \cap B$. We conclude that $A \cap B$ is sequentially compact.

Solution 2. Suppose that A is sequentially compact. This means that every sequence in A has a subsequence that converges to a point of A. Assume too that B is sequentially compact.

Our goal is to show that $A \cap B$ is sequentially compact. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of points in $A \cap B$. Since A is sequentially compact, by definition, there is some subsequence $(a_{n_i})_{i \in \mathbb{N}}$ converging to a point $a \in A$. Since B is sequentially compact and the subsequence $(a_{n_i})_{i \in \mathbb{N}}$ is contained in B, this subsequence must have a sub-subsequence that converges to a point $b \in B$.

However, by Proposition 1.2 on Worksheet #6, every subsequence of $(a_{n_i})_{i \in \mathbb{N}}$ must also converge to a, and so $a = b$. Thus the point a is contained in $A \cap B$. We proved that every sequence $(a_n)_{n \in \mathbb{N}}$ in $A \cap B$ has a subsequence converging to a point in $A \cap B$, hence we conclude that $A \cap B$ is sequentially compact as claimed.

2. Consider the metric space $X = (0, \infty) \subseteq \mathbb{R}$ with the Euclidean metric. For the following questions, simply state an example. **No justification needed.**

   (a) (1 point) Find a sequence $(a_n)_{n \in \mathbb{N}}$ in X that is Cauchy, and a continuous function $f : X \to X$ such that the sequence $(f(a_n))_{n \in \mathbb{N}}$ is not Cauchy.

   Example: Consider the sequence $\left(\frac{1}{n}\right)_{n \in \mathbb{N}}$, and the function $f(x) = \frac{1}{x}$. Then the sequence is Cauchy (since it converges to 0 in the larger metric space $\mathbb{R}$), but its image $(n)_{n \in \mathbb{N}}$ under f is not.

   (b) (1 point) Find a sequence $(a_n)_{n \in \mathbb{N}}$ in X that is unbounded, and a continuous function $f : X \to X$ such that the sequence $(f(a_n))_{n \in \mathbb{N}}$ is bounded.

   Example: Consider the sequence $(n)_{n \in \mathbb{N}}$, and the constant function $f(x) = 1$. Then the sequence is unbounded, but its image is the constant sequence $(1)_{n \in \mathbb{N}}$.