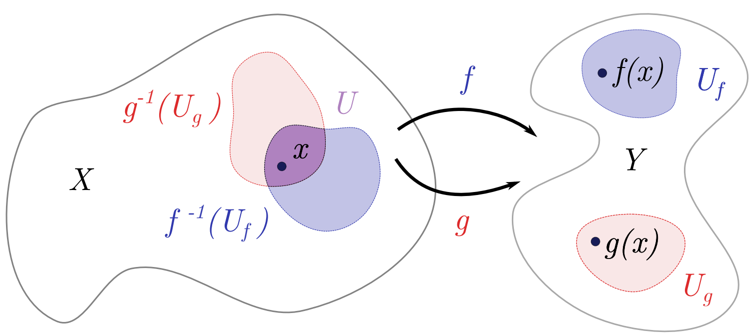
Name: ____________________ Score (Out of 5 points):

1. (5 points) Let $f, g : X \to Y$ be continuous maps between topological spaces. Show that if Y is Hausdorff, then the set $S = \{x \in X \mid f(x) = g(x)\}$ is closed.

Solution. To show that the set S is closed, we will show that its complement $X \setminus S$ is open. Let's call the complement Z. The set Z is, by definition, the set of points $x \in X$ such that $f(x) \neq g(x)$.

By Worksheet #9 Problem 3, to show Z is open, it suffices to show that every point of Z is an interior point of Z. (If Z is empty, there is nothing to check). So fix $x \in Z$. Our goal is to find a neighbourhood U of x such that $U \subseteq Z$.

Since $g(x) \neq f(x)$ and Y is Hausdorff, there exist disjoint neighbourhoods U_f of $f(x)$ and U_g of $g(x)$. Because the functions f and g are continuous, $f^{-1}(U_f)$ and $g^{-1}(U_g)$ are open subsets of X. Because $f(x) \in U_f$ and $g(x) \in U_g$, by definition of preimage, the point x must be contained in both $f^{-1}(U_f)$ and $g^{-1}(U_g)$. Thus, x is contained in the intersection $U = f^{-1}(U_f) \cap g^{-1}(U_g)$. Since U is the intersection of finitely many open sets, it is open.



We will show that U is contained in Z. Let $u \in U$. By our definition of U, $U \subseteq f^{-1}(U_f)$, hence $f(u) \in U_f$. Similarly $u \in U \subseteq g^{-1}(U_g)$, so $g(u) \in U_g$. But U_f and U_g are disjoint by construction, so we conclude that $f(u) \neq g(u)$. Thus $u \in Z$, and we conclude $U \subseteq Z$.

We have therefore proven that an arbitrary point $x \in Z$ has an open neighbourhood U contained in Z. We deduce that Z is open, and therefore that its complement S is closed.