

1 Continuous functions on topological spaces

Definition 1.1. (Continuous functions of topological spaces.) Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces. Then a map $f : X \rightarrow Y$ is called *continuous* if ...

In-class Exercises

1. Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces, and fix $y_0 \in Y$. Show that the constant map

$$\begin{aligned} f : X &\rightarrow Y \\ f(x) &= y_0 \end{aligned}$$

is continuous.

2. (a) Let X be a set, and let \mathcal{T}_\cdot denote the discrete topology on X . Let (Y, \mathcal{T}_Y) be a topological space. Show that any map $f : X \rightarrow Y$ of these topological spaces is continuous.
(b) Let (X, \mathcal{T}_X) be a topological space. Let Y be a set, and let \mathcal{T}_\bullet denote the indiscrete topology on Y . Show that any map $f : X \rightarrow Y$ of these topological spaces is continuous.
3. Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces, and let $S \subseteq X$. Let $f : X \rightarrow Y$ be a continuous function. Show that the restriction of f to S ,

$$f|_S : S \rightarrow Y,$$

is continuous with respect to the subspace topology on S .

4. Below are two results that you proved for metric spaces. Verify that each of these results holds for abstract topological spaces. This is a good opportunity to review their proofs!
- (a) **Theorem (Equivalent definition of continuity.)** Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces. Then a map $f : X \rightarrow Y$ is continuous if and only if for every closed set $C \subseteq Y$, the set $f^{-1}(C)$ is closed.
- (b) **Theorem (Composition of continuous functions.)** Let (X, \mathcal{T}_X) , (Y, \mathcal{T}_Y) , and (Z, \mathcal{T}_Z) be topological spaces. Suppose that $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are continuous maps. Prove that the map $g \circ f : X \rightarrow Z$ is continuous.

5. **(Optional).**

- (a) Let $f : X \rightarrow Y$ be a map of two topological spaces, and suppose that Y has the cofinite topology. Prove that f is continuous if and only if the preimage of every point in Y is closed in X .
- (b) Let $f : X \rightarrow Y$ be a map of two topological spaces that both have the cofinite topology. Prove that f is continuous if and only if either f is a constant function or the preimage of every point in Y is finite.

6. **(Optional).** Consider the following functions $f : \mathbb{R} \rightarrow \mathbb{R}$.

- (a) $f(x) = x$ (c) $f(x) = x^2$ (e) $f(x) = x + 1$
 (b) $f(x) = 0$ (d) $f(x) = \cos(x)$ (f) $f(x) = -x$

Determine whether these functions are continuous when both the domain and codomain \mathbb{R} have the topology ...

- standard (Euclidean) topology
- $\mathcal{T} = \{\mathbb{R}, \emptyset\}$
- $\mathcal{T} = \{\mathbb{R}, (0, 1), \emptyset\}$
- $\mathcal{T} = \{\mathbb{R}, \{0, 1\}, \{0\}, \{1\}, \emptyset\}$
- $\mathcal{T} = \{A \mid A \subseteq \mathbb{R}\}$
- cofinite topology
- $\mathcal{T} = \{(-\infty, a) \mid a \in \mathbb{R}\} \cup \{\emptyset\} \cup \{\mathbb{R}\}$
- $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\emptyset\} \cup \{\mathbb{R}\}$
- $\mathcal{T} = \{A \mid A \subseteq \mathbb{R}, 0 \in A\} \cup \{\emptyset\}$
- $\mathcal{T} = \{A \mid A \subseteq \mathbb{R}, 0 \notin A\} \cup \{\mathbb{R}\}$
- $\mathcal{T} = \{A \mid A \subseteq \mathbb{R}, 1 \in A\} \cup \{\emptyset\}$

7. **(Optional).** Let $X = \{a, b, c\}$ be the topological space with the topology

$$\{\emptyset, \{b\}, \{c\}, \{b, c\}, \{a, b, c\}\}.$$

Let \mathbb{R} be the topological space defined by the usual Euclidean metric. Which of the following functions $f : \mathbb{R} \rightarrow X$ are continuous?

- (i) $f(x) = b$ for all $x \in \mathbb{R}$.
- (ii) $f(x) = \begin{cases} a, & x \in (-\infty, 0) \\ b, & x = 0 \\ c, & x \in (0, \infty) \end{cases}$
- (iii) $f(x) = \begin{cases} a, & x = 0 \\ b, & x \in (-\infty, 0) \\ c, & x \in (0, \infty) \end{cases}$
- (iv) $f(x) = \begin{cases} a, & x \in (-\infty, 0] \\ b, & x \in (0, \infty) \end{cases}$

8. **(Optional).** Let $f : X \rightarrow Y$ be a function of topological spaces. Show that f is continuous if and only if it is continuous when restricted to be a map $f : X \rightarrow f(X)$, with $f(X) \subseteq Y$ given the subspace topology.9. **(Optional).** Let X be a set, and let \mathcal{T}_1 and \mathcal{T}_2 be topologies on X . Show that the identity map

$$\begin{aligned} id_X : (X, \mathcal{T}_1) &\rightarrow (X, \mathcal{T}_2) \\ id_X(x) &= x \end{aligned}$$

is continuous with respect to the topologies \mathcal{T}_1 and \mathcal{T}_2 if and only if \mathcal{T}_1 is finer than \mathcal{T}_2 .