1 Products of topological spaces

Definition 1.1. (The product topology.) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Then the *product topology* $\mathcal{T}_{X\times Y}$ on $X \times Y$ is the collection of subsets of $X \times Y$ generated by the set

$$\mathcal{B} = \{U \times V \mid U \subseteq X \text{ is open, and } V \subseteq Y \text{ is open}\}.$$

This means that $\mathcal{T}_{X\times Y}$ consists of all unions of elements of $\mathcal{B}$.

Proposition 1.2. *Let (X, d_X) and (Y, d_Y) be metric spaces. Then the product metric $d_{X\times Y}$ induces the product topology on $X \times Y$.*

In-class Exercises

1. Verify that $\mathcal{T}_{X\times Y}$ is indeed a topology on $X \times Y$, and that $\mathcal{B}$ is a basis for this topology.
   Hint: One approach is to use Worksheet #12 Problem 5.

2. Prove that the projection map $\pi_X : X \times Y \to X$ is both continuous and open with respect to the topologies $\mathcal{T}_{X\times Y}$ and $\mathcal{T}_X$. (The same argument shows that the projection map π_Y is both continuous and open).

3. Let $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$ be two functions of topological spaces, and define a function

$$f : X \to Y_1 \times Y_2$$
$$f(x) = (f_1(x), f_2(x))$$

   Show that f is continuous (with respect to the product topology on $Y_1 \times Y_2$) if and only if both f_1 and f_2 are continuous.
   Hint: Notice $f_i = \pi_i \circ f$, where $\pi_1 : Y_1 \times Y_2 \to Y_1$, $\pi_2 : Y_1 \times Y_2 \to Y_2$ are the projection maps.

4. **(Optional).** Prove the following theorem.

   Theorem (Equivalent definition of the product topology). Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Then the product topology $\mathcal{T}_{X\times Y}$ on $X\times Y$ is precisely the collection of subsets of $X \times Y$,

$$\mathcal{T}_{X\times Y} = \left\{ W \;\middle|\; \begin{array}{l} \text{for each } (x,y) \in W, \text{ there is a some } U \in \mathcal{T}_X \text{ and } V \in \mathcal{T}_Y \\ \text{such that } (x,y) \in (U \times V) \subseteq W \end{array} \right\}.$$

5. **(Optional). (Finite products).** Given a finite product $X = X_1 \times X_2 \times \cdots \times X_n$ of topological spaces, we can define a topology on X by induction, by first taking the product topology on $X_1 \times X_2$, then the product topology on $(X_1 \times X_2) \times X_3$, etc. Show that the resultant topology on X (called the *product topology*) is generated by the basis
$$\mathcal{B} = \{U_1 \times U_2 \times \cdots \times U_n \mid U_i \subseteq X_i \text{ is open}\}.$$

6. **(Optional). (Infinite products).** Let $\{X_i\}_{i \in I}$ be a (possibly infinite) collection of sets, and let $X = \prod_{i \in I} X_i$ be their product. We denote elements of X by $(x_i)_{i \in I}$. Define two topologies on X:

   - The *box topology* on X is the topology generated by the basis
   $$\mathcal{B}_B = \left\{ \prod_{i \in I} U_i \;\middle|\; U_i \subseteq X_i \text{ is open} \right\}.$$
   - The *product topology* on X is the topology generated by the basis
   $$\mathcal{B}_P = \left\{ \prod_{i \in I} U_i \;\middle|\; U_i \subseteq X_i \text{ is open, } U_i = X \text{ for all but finitely many } i \in I \right\}.$$

   (a) Prove that both $\mathcal{B}_B$ and $\mathcal{B}_P$ are bases (in the sense of Worksheet #12, Problem 5). Conclude that the box topology and product topology on X are, in fact, topologies.

   (b) Suppose that I is finite. Show that the box topology and the product topology are equal, and both are the usual product topology on X in the sense of Problem 5.

   (c) Which topology is finer, the box or the product topology? If we consider maps from a topological space into the product X, what can you say about the relationship between continuity of a map with respect to the box topology, and continuity with respect to the product topology? What about for maps out of X?

   (d) For reasons that are formalized using "category theory" and the concept of a "universal property", we want our products to satisfy the following statement:

   > Let $\{Y_i\}_{i \in I}$ be a collection of topological spaces, and $\prod_{i \in I} Y_i$ their product. Let X be any topological space, and let $f_i : X \to Y_i$ be a collection of functions. Then the function
   > $$f : X \to \prod_{i \in I} Y_i$$
   > $$f(x) = (f_i(x))_{i \in I}$$
   > is continuous if and only if each function f_i is continuous.

   Prove that this property always holds if we put the product topology on $\prod_{i \in I} Y_i$, but that this property may fail if we put the box topology on $\prod_{i \in I} Y_i$. This property is the reason that the product topology is generally considered the "correct" topology on $\prod_{i \in I} Y_i$. *Hint:* Consider the function $f : \mathbb{R} \to \prod_{n \in \mathbb{N}} \mathbb{R}$ given by $f(x) = (x, x, x, \ldots)$.

   (e) Let $\{X_n\}_{n \in \mathbb{N}}$ be a countable collection of metric spaces. Show that the product topology on $\prod_{n \in \mathbb{N}} X_n$ is metrizable, but the box topology is not.