

## 1 Sequences in topological spaces

**Definition 1.1. (Convergence topological spaces.)** Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence of points in a topological space  $(X, \mathcal{T})$ . Then we say that  $(a_n)_{n \in \mathbb{N}}$  *converges* to a point  $a_\infty \in X$  if ...

**Example 1.2.** Let  $X = \{0, 1\}$  with the topology  $\mathcal{T} = \{\emptyset, \{0\}, \{0, 1\}\}$ . Find all limits of the following sequences.

(a) Constant sequence  $0, 0, 0, 0, 0, \dots$

(b) Constant sequence  $1, 1, 1, 1, 1, \dots$

(c) Alternating sequence  $0, 1, 0, 1, 0, \dots$

**Example 1.3.** Let  $X = \mathbb{R}$  with the cofinite topology. Find all limits of the sequence  $(n)_{n \in \mathbb{N}}$ .

### In-class Exercises

- Let  $X$  be a topological space, and let  $x \in X$ . Show that the constant sequence  $a_n = x$  converges to  $x$ . Could it also converge to other points of  $X$ ?
- Give an example of a topological space  $(X, \mathcal{T})$ , and a sequence  $(a_n)_{n \in \mathbb{N}}$  in  $X$  that converges to (at least) two distinct points  $a_\infty \in X$  and  $\tilde{a}_\infty \in X$ .
  - Now suppose that  $(X, \mathcal{T})$  is a **Hausdorff** topological space, and let  $(a_n)_{n \in \mathbb{N}}$  be a sequence in  $X$ . Show that, if  $(a_n)_{n \in \mathbb{N}}$  converges, then it converges to only one point  $a_\infty$ .
- Definition (Sequential continuity).** Let  $f : X \rightarrow Y$  be a function of topological spaces. Then  $f$  is called *sequentially continuous* if for any sequence  $(x_n)_{n \in \mathbb{N}}$  in  $X$  and any limit  $x_\infty$  of the sequence, the sequence  $(f(x_n))_{n \in \mathbb{N}}$  in  $Y$  converges to the point  $f(x_\infty)$ .

You proved that on Homework #3 Problem 4 that if  $X$  and  $Y$  are metric spaces (or, more generally, metrizable topological spaces), then sequential continuity is equivalent to continuity for a function  $f : X \rightarrow Y$ .

Prove that any continuous function  $f : X \rightarrow Y$  of topological spaces is sequentially continuous. It turns out that the converse does not hold in general for abstract topological spaces!

- Let  $A$  be a subset of a topological space  $(X, \mathcal{T})$ . Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence of points in  $A$  that converge to a point  $a_\infty \in X$ . Prove that  $a_\infty \in \overline{A}$ .

5. **(Optional)**. For each of the following sequences: find the set of all limits, or determine that the sequence does not converge.

- Let  $X = \{a, b, c, d\}$  have the topology  $\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c, d\}\}$ .
  - (i)  $a, b, a, b, a, b, a, b, \dots$
- Let  $\mathbb{R}$  have the topology  $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\emptyset\} \cup \{\mathbb{R}\}$ .
  - (ii)  $0, 0, 0, 0, 0, 0, \dots$
  - (iii)  $(n)_{n \in \mathbb{N}}$
  - (iv)  $(-n)_{n \in \mathbb{N}}$
- Let  $\mathbb{R}$  have the topology  $\mathcal{T} = \{\emptyset\} \cup \{U \subseteq \mathbb{R} \mid 0 \in U\}$ .
  - (v)  $0, 0, 0, 0, 0, 0, \dots$
  - (vi)  $1, 1, 1, 1, 1, 1, \dots$

6. **(Optional)**. Let  $X = \{0, 1\}$  with the topology  $\mathcal{T} = \{\emptyset, \{0\}, \{0, 1\}\}$ . Prove that every sequence of points in  $X$  converges to 1.

7. **(Optional)**. In this problem, we will construct an example of a function of topological spaces that is sequentially continuous but not continuous.

- (a) Recall that a set  $S$  is *countable* if there exists an injective function  $S \rightarrow \mathbb{N}$ , equivalently, if there is a surjective functions  $\mathbb{N} \rightarrow S$ . Such sets are either finite or countably infinite. Define a topology on  $\mathbb{R}$  by

$$\mathcal{T}_{cc} = \{\emptyset\} \cup \{U \subseteq \mathbb{R} \mid \mathbb{R} \setminus U \text{ is countable}\}.$$

Show that  $\mathcal{T}_{cc}$  is indeed a topology on  $\mathbb{R}$ . It is called the *co-countable topology*.

- (b) Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence in  $(\mathbb{R}, \mathcal{T}_{cc})$ . Show that  $(a_n)_{n \in \mathbb{N}}$  converges if and only if it is *eventually constant*. This means there is some  $N \in \mathbb{N}$  and  $x \in \mathbb{R}$  so that  $a_n = x$  for all  $n \geq N$ .
- (c) Let  $\mathcal{T}_{dsc}$  denote the discrete topology on  $\mathbb{R}$ . Let  $I : \mathbb{R} \rightarrow \mathbb{R}$  be the identity map. Show that the following map of topological spaces is **not** continuous:

$$I : (\mathbb{R}, \mathcal{T}_{cc}) \rightarrow (\mathbb{R}, \mathcal{T}_{dsc})$$

- (d) Show that  $I : (\mathbb{R}, \mathcal{T}_{cc}) \rightarrow (\mathbb{R}, \mathcal{T}_{dsc})$  is sequentially continuous.

8. **(Optional)**.

- (a) Suppose  $(X, d)$  is a metric space, and  $A \subseteq X$ . Prove that  $x \in \overline{A}$  if and only if there is some sequence of points  $(a_n)_{n \in \mathbb{N}}$  in  $A$  that converge to  $x$ .
- (b) Consider  $\mathbb{R}$  with the cocountable topology, and let  $A \subseteq \mathbb{R}$ . What is  $\overline{A}$  if  $A$  is (i) countable, or (ii) uncountable?
- (c) Let  $A = (0, 1)$ , so  $\overline{A} = \mathbb{R}$ . Show that, for any  $x \in \overline{A} \setminus A$ , there is **no** sequence of points in  $A$  that converge to  $x$ .
- (d) **Definition (First countable spaces)**. A topological space  $(X, \mathcal{T})$  is called *first countable* if each point  $x \in X$  has a *countable neighbourhood basis*. This means, for each  $x \in X$ , there is a countable collection  $\{N_i\}_{i \in \mathbb{N}}$  of neighbourhoods of  $x$  with the property that, if  $N$  is any neighbourhood of  $x$ , then there is some  $i$  such that  $N_i \subseteq N$ .

Let  $X$  be a first countable space, and let  $A \subseteq X$ . Show that, given any  $x \in \overline{A}$ , there is some sequence of points in  $A$  that converges to  $x$ .