

1 Connected and path-connected topological spaces

Definition 1.1. (Path-connected spaces). Let (X, \mathcal{T}) be a topological space, and let $x, y \in X$. Recall that a *path from x to y* is a continuous function $\gamma : ([0, 1], \text{Euclidean}) \rightarrow (X, \mathcal{T})$ such that $\gamma(0) = x$ and $\gamma(1) = y$. A space (X, \mathcal{T}) is *path-connected* if, given any two points $x, y \in X$, there exists a path from x to y .

Example 1.2. Sierpiński space is the space $\mathbb{S} = \{0, 1\}$ with the topology $\mathcal{T} = \{\emptyset, \{0\}, \{0, 1\}\}$. Is \mathbb{S} path-connected?

In-class Exercises

- In this problem, we will prove the following result:

Theorem (Connectivity of product spaces). Let X and Y be nonempty topological spaces. Then the product space $X \times Y$ (with the product topology) is connected if and only if both X and Y are connected.

Hint: See Worksheet #15, Problem 6(b).

- Suppose that $X \times Y$ is nonempty and connected in the product topology $\mathcal{T}_{X \times Y}$. Prove that X and Y are connected.
 - Suppose that (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) are nonempty, connected spaces, and suppose that $(a, b) \in X \times Y$. Prove that $(X \times \{b\}) \cup (\{a\} \times Y)$ is a connected subset of the product $X \times Y$ with the product topology $\mathcal{T}_{X \times Y}$.
 - Suppose that (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) are nonempty, connected spaces. Prove that $X \times Y$ is connected in the product topology $\mathcal{T}_{X \times Y}$.
- Let X be a (nonempty) topological space with the indiscrete topology. Is X necessarily path-connected?
 - Prove the following.

Theorem (Path-connected \implies connected). Let (X, \mathcal{T}) be a topological space. If X is path-connected, then X is connected.

Hint: You may use the result from Homework #11 that the interval $[0, 1]$ is connected.

- Let $f : X \rightarrow Y$ be a continuous map of topological spaces. Prove that if X is path-connected, then $f(X)$ is path-connected. In other words, the continuous image of a path-connected space is path-connected.
- (Optional).** Recall that $\mathcal{C}(0, 1)$ denotes the set of continuous functions from the closed interval $[0, 1]$ to \mathbb{R} , and that $\mathcal{C}(0, 1)$ is a metric space with metric

$$d_\infty : \mathcal{C}(0, 1) \times \mathcal{C}(0, 1) \longrightarrow \mathbb{R}$$

$$d(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)|$$

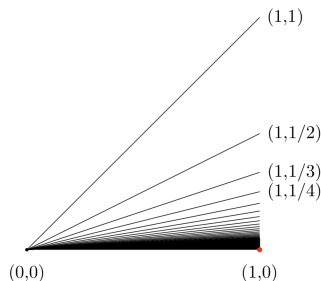
Show that this metric space is path-connected, and therefore connected.

6. **(Optional)**. This problem shows that the converse to Problem 3 fails.

Let X be the following subspace of \mathbb{R}^2 (with topology induced by the Euclidean metric)

$$X = \{(1, 0)\} \cup \bigcup_{n \in \mathbb{N}} L_n,$$

where L_n is the closed line segment connecting the origin $(0, 0)$ to the point $(1, \frac{1}{n})$.



(a) Show that X is connected.

(b) **(Challenge)**. Show that X is **not** path-connected.

(c) Would the space be path-connected if we added in the line segment from $(0, 0)$ to $(1, 0)$?

7. **(Optional)**.

Definition (Local connectedness). Let (X, \mathcal{T}) be a topological space. Then X is *locally connected at a point* $x \in X$ if every neighbourhood U_x of x contains a connected open neighbourhood V_x of x . The space X is *locally connected* if it is locally connected at every point $x \in X$.

Definition (Local path-connectedness). Let (X, \mathcal{T}) be a topological space. Then X is *locally connected at a point* $x \in X$ if every neighbourhood U_x of x contains a path-connected open neighbourhood V_x of x . The space X is *locally path-connected* if it is locally path-connected at every point $x \in X$.

- (a) Let (X, \mathcal{T}) be a topological space, and let $x \in X$. Show that if X is locally path-connected at x , then it is locally connected at x . Conclude that locally path-connected spaces are locally connected.
- (b) Let $X = (0, 1) \cup (2, 3)$ with the Euclidean metric. Show that X is locally path-connected and locally connected, but is not path-connected or connected.
- (c) Let X be the following subspace of \mathbb{R}^2 (with topology induced by the Euclidean metric)

$$X = \bigcup_{n \in \mathbb{N}} \left(\left\{ \frac{1}{n} \right\} \times [0, 1] \right) \cup \left(\{0\} \times [0, 1] \right) \cup \left([0, 1] \times \{0\} \right).$$

Show that X is path-connected and connected, but not locally connected or locally path-connected.

- (d) **(Challenge)**. Consider the natural numbers \mathbb{N} with the cofinite topology. Show that \mathbb{N} is locally connected but not locally path-connected.