1 The interior and the closure of a set

Definition 1.1. (Interior of a set.) Let (X, d) be a metric space, and $A \subseteq X$ a subset. Then the *interior of* A, denoted $\text{Int}(A)$ or $\mathring{A}$, is defined to be the set

$$\text{Int}(A) = \{a \in A \mid a \text{ is an interior point of } A\}.$$

Note that $\text{Int}(A) \subseteq A$. We will see in the exercises that $\text{Int}(A)$ is an open set, and it is in a sense the largest open subset of A.

Definition 1.2. (Closure of a set.) Let (X, d) be a metric space, and $A \subseteq X$ a subset. Then the *closure of* A, denoted $\overline{A}$, is defined to be the set

$$\overline{A} = \{x \in X \mid \text{ for every } r > 0 \text{ the ball } B_r(x) \text{ contains a point of } A\}.$$

We will see that $\overline{A}$ is a closed set, and that in a sense it is the smallest closed set containing A.

Example 1.3. What is the closure of the open set $B_1(0, 0) \subseteq \mathbb{R}^2$?

In-class Exercises

1. For this problem we introduce the following terminology.

   Definition 1.4. (Neighbourhood of a point x.) Let (X, d) be a metric space, and $x \in X$. Then any open set U containing x is called an *open neighbourhood of x*, or simply a *neighbourhood of x*.

   (a) Prove the following.
   Theorem 1.5. (Equivalent definition of interior point.) *For a subset V of a metric space X, a point $x \in V$ is an interior point of V if and only if there exists an open neighbourhood U of x that is contained in V.*

   (b) Prove the following.
   Theorem 1.6. (Equivalent definition of closure.) *For a subset A of a metric space X, the closure of A is equal to the set*

   $$\overline{A} = \{x \in X \mid \textit{every neighbourhood } U \textit{ of } x \textit{ contains a point of } A\}.$$

2. Prove the following theorem.

Theorem 1.7. *Let (X, d) be a metric space, and $A \subseteq X$ a subset.*

(i) $\text{Int}(A) \subseteq A$
(ii) *A is open if and only if* $A = \text{Int}(A)$
(iii) *If* $A \subseteq B$ *then* $\text{Int}(A) \subseteq \text{Int}(B)$
(iv) $\text{Int}(\text{Int}(A)) = \text{Int}(A)$
(v) $\text{Int}(A)$ *is open in X*
(vi) $\text{Int}(A)$ *is the largest open subset of A in the following sense: If* $U \subseteq A$ *is any open subset of A, then* $U \subseteq \text{Int}(A)$

3. Prove the following theorem.

Theorem 1.8. *Let (X, d) be a metric space, and $A \subseteq X$ a subset.*

(i) $A \subseteq \overline{A}$
(ii) *If* $A \subseteq B$ *then* $\overline{A} \subseteq \overline{B}$
(iii) *A is closed if and only if* $A = \overline{A}$
(iv) $\overline{\overline{A}} = \overline{A}$
(v) $\overline{A}$ *is closed in X*
(vi) $\overline{A}$ *is the smallest closed set containing A, in the following sense: If* $A \subseteq C$ *for some closed set C, then* $\overline{A} \subseteq C$

4. **(Optional).** Let A be a subset of a metric space (X, d). Explore the relationships between the sets

$$\text{Int}(X \setminus A) \qquad X \setminus \text{Int}(A) \qquad \overline{X \setminus A} \qquad X \setminus \overline{A}$$

Determine which of these sets are necessarily equal or necessarily subsets of one another. Give counterexamples to show where equality or containment fails.

5. **(Optional).** Let A_i, $i \in I$, be a collection of subsets of a metric space (X, d). For each of the following statements, either prove the statement, or construct a counterexample.

(a) $\text{Int}\left(\bigcup_{i \in I} A_i\right) \subseteq \bigcup_{i \in I} \text{Int}(A_i)$

(b) $\text{Int}\left(\bigcup_{i \in I} A_i\right) \supseteq \bigcup_{i \in I} \text{Int} A_i$

(c) $\text{Int}\left(\bigcap_{i \in I} A_i\right) \subseteq \bigcap_{i \in I} \text{Int}(A_i)$

(d) $\text{Int}\left(\bigcap_{i \in I} A_i\right) \supseteq \bigcap_{i \in I} \text{Int}(A_i)$

(e) $\overline{\bigcup_{i \in I} A_i} \subseteq \bigcup_{i \in I} \overline{A_i}$

(f) $\overline{\bigcup_{i \in I} A_i} \supseteq \bigcup_{i \in I} \overline{A_i}$

(g) $\overline{\bigcap_{i \in I} A_i} \subseteq \bigcap_{i \in I} \overline{A_i}$

(h) $\overline{\bigcap_{i \in I} A_i} \supseteq \bigcap_{i \in I} \overline{A_i}$

6. **(Optional).** Prove the following equivalent definition of continuity.

Theorem (An equivalent definition of continuity). Let (X, d_X) and (Y, d_Y) be metric spaces. Then a map $f : X \to Y$ is continuous if and only if

$$f\left(\overline{A}\right) \subseteq \overline{f(A)} \qquad \text{for every subset } A \subseteq X.$$

7. **(Optional).** For a metric (X, d), let $x_0 \in X$ and $r > 0$. You proved on the homework that the set

$$C_r(x_0) = \{x \in X \mid d(x, x_0) \leq r\}$$

is closed. Explain why $C_r(x_0)$ always contains the closure of the ball $B_r(x_0)$. Give an example of a metric space where $C_r(x_0)$ is equal to $\overline{B_r(x_0)}$ for every $r > 0$ and x_0, and give an example of a metric space and x_0, r such that $C_r(x_0)$ is a strict subset of $\overline{B_r(x_0)}$.