

1 Sequential Compactness

Definition 1.1. (Subsequences.) Let (X, d) be a metric space, and let $(a_n)_{n \in \mathbb{N}}$ be a sequence in X . Let

$$0 < n_1 < n_2 < \cdots < n_i < \cdots$$

be an infinite sequence of strictly increasing natural numbers. Then $(a_{n_i})_{i \in \mathbb{N}}$ is called a *subsequence* of $(a_n)_{n \in \mathbb{N}}$.

Proposition 1.2. *Suppose that $(a_n)_{n \in \mathbb{N}}$ is a sequence in a metric space (X, d) that converges to a point a_∞ . Show that any subsequence of $(a_n)_{n \in \mathbb{N}}$ also converges to a_∞ .*

Definition 1.3. (Sequentially compact metric spaces; sequentially compact subsets.) A metric space (X, d) is called *sequentially compact* if every sequence in X has a convergent subsequence. Similarly, a subset $S \subseteq X$ is *sequentially compact* if every sequence of points in S has a subsequence that converges to a point in S .

The empty set \emptyset is sequentially compact.

Example 1.4. Give examples of sequences $(a_n)_{n \in \mathbb{N}}$ of real numbers with the following properties:

- (a) The sequence $(a_n)_{n \in \mathbb{N}}$ has a subsequence that converges to 0, and a subsequence that converges to 1.

- (b) The sequence $(a_n)_{n \in \mathbb{N}}$ has no convergent subsequence.

- (c) The sequence $(a_n)_{n \in \mathbb{N}}$ is an unbounded sequence with a subsequence that converges to 0.

In-class Exercises

- Suppose that (X, d) is a sequentially compact metric space, and that $C \subseteq X$ is a closed subset. Prove that C is sequentially compact.
- Let (X, d) be a metric space.
 - Prove that every sequentially compact subset of X is closed.
 - Definition (Bounded subset.)** Let (X, d) be a metric space. A subset $S \subseteq X$ is called *bounded* if there is some $x_0 \in X$ and some $R \in \mathbb{R}$ with $R > 0$ such that $S \subseteq B_R(x_0)$.
Prove that every sequentially compact subset of X is bounded.
 - Is it true that every closed and bounded subset of a metric space is sequentially compact?
- (Optional).** Let (X, d) be a metric space, and $A \subseteq X$ a subset. Show that A is bounded if and only if there is some $M \in \mathbb{R}$ so that $d(a, b) \leq M$ for all $a, b \in A$.
- (Optional).** Let X be a metric space with the discrete metric. Give necessary and sufficient conditions for a subset of X to be sequentially compact.
- (Optional).** Recall that the space $\mathcal{C}([a, b])$ of continuous functions $f : [a, b] \rightarrow \mathbb{R}$ is a metric space with metric

$$d_\infty(f, g) = \sup_{x \in [a, b]} |f(x) - g(x)|.$$

- Show that $\mathcal{C}([a, b])$ is unbounded. Conclude that it is **not** sequentially compact.
- Consider the subset

$$C = \{f(x) \in \mathcal{C}([0, 1]) \mid |f(x)| \leq 1 \text{ for all } x \in [0, 1]\}.$$

Is C sequentially compact?

- (Optional).** Let X be a metric space, and let $a_\infty \in X$. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence in X with the property that every subsequence of $(a_n)_{n \in \mathbb{N}}$ has a subsequence that converges to a_∞ . Prove that $(a_n)_{n \in \mathbb{N}}$ converges to a_∞ .
- (Optional).**
 - Consider the sequence of real numbers

$$0, 1, 0, \frac{1}{2}, 1, 0, \frac{1}{3}, \frac{2}{3}, 1, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, 0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1, \dots$$

What real numbers can be realized as the limit of a subsequence of this sequence?

- Is it possible to construct a sequence $(a_n)_{n \in \mathbb{N}}$ with subsequences converging to every real number $r \in \mathbb{R}$?