1 Product Metrics

Definition 1.1. Let (X, d_X) and (Y, d_Y) be metric spaces. Then their Cartesian product $X \times Y$ has a metric space structure, defined by the metric

$$\begin{aligned} d_{X\times Y} :\ & (X \times Y) \times (X \times Y) \longrightarrow \mathbb{R} \\ d_{X\times Y}\Big((x_1, y_1), (x_2, y_2)\Big) &= \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}. \end{aligned}$$

We will call $d_{X\times Y}$ the *product metric* on $X \times Y$.[1]

Example 1.2. Consider $\mathbb{R}$ with the Euclidean metric. What is the product metric on $\mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$?

In-class Exercises

1. Verify that $d_{X\times Y}$ does in fact define a metric on $X \times Y$.
2. (a) Prove that if $U \subseteq X$ and $V \subseteq Y$ are open sets, then $U \times V$ is an open subset of $X \times Y$.

   (b) Let $U \subseteq X \times Y$ be an open set, and let $(x, y) \in U$. Show that there is a neighbourhood U_x of x in X and a neighbourhood U_y of y in Y so that $U_x \times U_y \subseteq U$.
3. **Definition (Projection maps).** For a product of sets $X \times Y$, the maps

$$\begin{aligned} \pi_X &: X \times Y \to X & \qquad \pi_Y &: X \times Y \to Y \\ \pi_X(x, y) &= x & \pi_Y(x, y) &= y \end{aligned}$$

   are called the *projection onto* X and the *projection onto* Y, respectively.

   (a) Let (X, d_X) and (Y, d_Y) be metric spaces, and endow their product $X\times Y$ with the product metric $d_{X\times Y}$. Show that the projection map

$$\pi_X : (X \times Y, d_{X\times Y}) \longrightarrow (X, d_X)$$

   is continuous. (The same argument, which you do not need to repeat, shows that the map π_Y is continuous).

   (b) **Definition (Open maps).** A map $f : W \to Z$ of metric spaces is called *open* if $f(U) \subseteq Z$ is open for every open subset $U \subseteq W$. In other words, the image of every open subset is open.

   Prove that the projection map π_X is open. (The same argument shows π_Y is open).

   Hint: Use Question 2.

[1] *Remark:* In light of Problem 6(b), any of the metrics in Problem 6 – among others – are sometimes called the *product metrics* on $X \times Y$. For clarity in this course we will use this term specifically for the metric $d_{X\times Y}$.

4. **(Optional).** Compare and contrast the definition of an **open** map with the definition of a **continuous** map. Find examples of maps $f : X \to Y$ that are ...

   (a) open but not continuous
   (b) continuous but not open
   (c) open and continuous
   (d) neither continuous nor open

5. **(Optional).** A map is called *closed* if the image of every closed set is closed. Prove or find a counterexample: the projection map $\pi_X : X \times Y \to X$ is always a closed map.

6. **(Optional).** Let (X, d_X) and (Y, d_Y) be metric spaces.

   (a) Show that each of the following functions also defines a metric on $X \times Y$.

$$d_1 : \; (X \times Y) \times (X \times Y) \longrightarrow \mathbb{R}$$
$$d_1((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2).$$

$$d_\infty : \; (X \times Y) \times (X \times Y) \longrightarrow \mathbb{R}$$
$$d_{X\times Y}((x_1, y_1), (x_2, y_2)) = \max\left\{d_X(x_1, x_2), d_Y(y_1, y_2)\right\}.$$

   (b) Show that these metrics on $X \times Y$ are all *topologically equivalent* to the metric $d_{X\times Y}$. This means that a subset of $X \times Y$ open with respect to one metric if and only if it is open with respect to the other.

7. **(Optional)**. Let (X_i, d_i) be metric spaces for $i \in \mathbb{N}$. Can you find a natural way to define a metric on the infinite product $\prod_{i\in\mathbb{N}} X_i = X_1 \times X_2 \times X_3 \times \cdots$?
   There are multiple ways to do this, which are not topologically equivalent!

8. **(Optional).** Let (X, d) be a metric space. Show that the map $d : X \times X \to \mathbb{R}$ is continuous with respect to the product metric on $X \times X$ and the standard topology on $\mathbb{R}$.