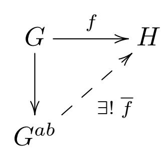
The following questions are intended to help students assess whether they have an appropriate level of background for Math 592. I expect that students in the course should have the background necessary to complete this 'homework assignment' independently and with minimal-to-no use of outside references. **If you have difficulty with any of these questions (except the bonus) please speak to Jenny about whether to take Math 592.**

I do not necessarily expect students to be already familiar with solutions to all of these problems, or to know how to solve them all immediately. In fact, I may assign variations on some of these problems as homework during the term. But I do expect that students already have the preparation they need to solve the problems; I would assign these questions without covering any additional background in lecture. I expect that students entering the class should be comfortable doing this assigment as they would a regular homework set.

A warning: if you do not have the recommended prerequisites in algebra or topology, then I would not expect that learning how to solve these specific questions would be sufficient preparation for the course. These questions cover only a sampling of background topics. They are intended to be diagnostic, not exhaustive.

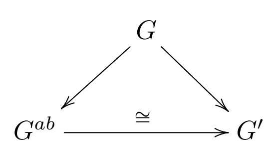
Assignment Questions

1. **(Abelianization).** Let G be a group. We define the *commutator subgroup* $[G, G]$ of G to be the subgroup normally generated by by *commutators*, elements of the form $ghg^{-1}h^{-1}$ for all $g, h \in G$. We define the *abelianization* G^{ab} of G to be the quotient group $G/[G, G]$.
   (a) Show that G^{ab} is an abelian group.
   (b) Show moreover that if G is abelian, then $G = G^{ab}$.
   (c) Show that the quotient map $G \to G^{ab}$ satisfies the following universal property: Given any **abelian** group H and group homomorphism $f : G \to H$, there is a unique group homomorphism $\overline{f} : G^{ab} \to H$ that makes the following diagram commute:



   This universal property shows that G^{ab} is in a sense the "largest" abelian quotient of G.

   (d) Show that G^{ab} is uniquely defined by the universal property, in the following sense: if any other abelian group G', equipped with a distinguished map $G \to G'$, satisfied this property, there would be an isomorphism of groups $G' \cong G^{ab}$, and this would be the unique isomorphism making the following diagram commute:



   Your proof should just use the formal statement of the universal property, and not particular properties of groups or their commutator subgroups. You should be able to use this same proof to prove the following principle for any universal property:

   > "A universal property defines its object uniquely up to unique isomorphism."

   Conclude that the universal property could, in fact, be taken to be the *definition* of the abelianization of G.

   (e) Let H be an abelian group. Use this universal property to construct a bijection between the set of group homomorphisms $\mathrm{Hom}(G, H)$ and the set of group homomorphisms $\mathrm{Hom}(G^{ab}, H)$.

Remark. It turns out this bijection is *natural* – a technical category theoretic term we will not define, but that means the bijection respects the group structures of G and H in some sense. The existence of this natural bijection is the statement that "abelianization" (viewed as a functor from the category of groups to the category of abelian groups) is an adjoint functor to the inclusion functor from the category of abelian groups to the category of groups. None of this remark needs to make sense at this point!

2. **(The free group on a set).** Let S be a set. We will construct the *free group* $F(S)$ on S, as follows. A *word* w is a finite sequence (possibly empty) in the formal symbols $\{s, s^{-1} \mid s \in S\}$. Then (as a set) we define $F(S)$ to be equivalence classes of words under the equivalence relation

$$vs^{-1}sw \sim vw \qquad \text{and} \qquad vss^{-1}w \sim vw \qquad \text{for any words } w, v.$$

Given equivalence classes $[w]$ and $[v]$, we define the group operation by concatenation of words:

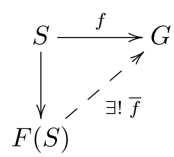
$$[v] \cdot [w] = [vw].$$

(a) Verify that the group multiplication is well-defined, and that $F(S)$ is in fact a group.

(b) For a set S, consider the free group $F(S)$. Observe that there is a canonical inclusion of sets

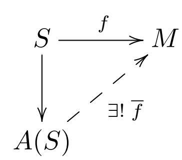
$$S \longrightarrow F(S)$$
$$s \longmapsto [s]$$

Show that $F(S)$ satisfies the following universal property: given any group G and any map of sets $f : S \to G$, the map f extends uniquely to a group homomorphism $\overline{f} : F(S) \to G$. In other words, there is a unique homomorphism $\overline{f}$ making the following diagram commute.



(c) Let G be a group and $\iota : S \hookrightarrow G$ the inclusion of a subset S of G. Show that S generates G if and only if the homomorphism $F(S) \to G$ extending ι is surjective.

(d) Let S be a set. The *free abelian group* $A(S)$ on S is an abelian group defined by the following universal property. There is a map $S \to A(S)$. Given any other abelian group M, then any map of sets $f : S \to M$ extends uniquely to a group homomorphism $\overline{f} : A(S) \to M$. In other words, there is a unique group map $\overline{f}$ making the following diagram commute.



Show that the abelianization of $F(S)$ (in the sense of Problem 1) is the free abelian group on S, by verifying that it satisfies the defining universal property.

3. **(Homomorphisms of free abelian groups).** Let A be an $n \times n$ integer matrix, viewed as $\mathbb{Z}$-linear map from $\mathbb{Z}^n$ to $\mathbb{Z}^n$.

(a) Suppose that A has rank n. Prove that the kernel of A is trivial.
(Note: Here we mean 'rank' in the usual sense from linear algebra, for example, it is the rank of A when A is viewed as a matrix with entries in $\mathbb{Q}$).

(b) Show by example that, even if A has rank n, it need not be surjective.

(c) The *cokernel* of a map of abelian groups is the quotient of its codomain by its image. Prove or find a counterexample: if the map A has rank n, then the cokernel of A must be finite.

4. **(The splitting lemma).** Let R be a ring. A *short exact sequence* of R-modules is a sequence of R-module homomorphisms

$$0 \longrightarrow N \xrightarrow{\varphi} M \xrightarrow{\psi} Q \longrightarrow 0$$

that is *exact* at N, M, and Q. This means that the image of the incoming map is equal to the kernel of the outgoing map. In other words, φ is injective, ψ is surjective, and $\mathrm{Im}(\varphi) = \mathrm{Ker}(\psi)$.

(a) Prove that the following three conditions are equivalent. A short exact sequence satisfying these conditions is said to *split*.

(i) There exists a map $\psi' : Q \to M$ such that $\psi \circ \psi'$ is the identity on Q.

(ii) There exists a map $\varphi' : M \to N$ such that $\varphi' \circ \varphi$ is the identity on N.

(iii) There is a direct sum decomposition $M \cong N' \oplus Q'$ such that φ defines an isomorphism $N \xrightarrow{\cong} N' = \varphi(N)$ and ψ restricts to an isomorphism $\psi|_{Q'} : Q' \xrightarrow{\cong} Q$.

(b) Determine which of the following short exact sequences of abelian groups split.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 5} \mathbb{Z} \xrightarrow{\text{mod } 5} \mathbb{Z}/5\mathbb{Z} \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z}/3\mathbb{Z} \xrightarrow{\times 5} \mathbb{Z}/15\mathbb{Z} \xrightarrow{\text{mod } 5} \mathbb{Z}/5\mathbb{Z} \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 1 & -1 \end{bmatrix}} \mathbb{Z} \longrightarrow 0$$

5. **(Connected components and continuity).** Either prove the following statement, or give (with proof) a counterexample:

> Let X, Y be topological spaces and $f : X \to Y$ a map. Then f is continuous if and only if its restriction to each connected component is continuous.

6. **(Orbits and stabilizers).** For a group G, a *G-set* X is a set X with an action of G. For G-sets X and Y, a *map of G-sets* is a map $f : X \to Y$ of sets that commutes with the group action, that is,

$$g \cdot f(x) = f(g \cdot x) \qquad \text{for all } g \text{ in } G \text{ and } x \text{ in } X.$$

The map f is an *isomorphism of G-sets* if, in addition, it is bijective.

(a) Let X be a G-set. Suppose that $x_1, x_2 \in X$ are two points in the same G-orbit. Construct an explicit isomorphism between the stabilizer G_{x_1} of x_1 and the stabilizer G_{x_2} of x_2.

(b) Suppose a group G acts transitively on a set X. Show that there is an isomorphism of G-sets $X \cong G/G_x$, where G_x is a the stabilizer of a point $x \in X$, and G acts on the cosets G/G_x by left multiplication.

(c) Let G and H be two groups acting on a set X. Assume that the actions *commute* in the sense that $g \cdot (h \cdot x) = h \cdot (g \cdot x)$ for all $g \in G, h \in H$, and $x \in X$. Let Y be the subset of X that is fixed pointwise by G. Show that Y is stabilized by H. Must Y be fixed pointwise by H?

7. **(Quotient topologies).** Let I be the closed interval $[0, 1]$, and let S^1 be the unit circle in $\mathbb{C}$. Let D^2 be the closed unit ball in $\mathbb{C}$.

(a) Prove the following theorem.

> **Theorem.** Let $f : X \to Y$ be a continuous, bijective map of topological spaces. Suppose X is compact and Y is Hausdorff. Then f is a homeomorphism.

(b) Consider the cylinder $I \times S^1$. Let D be the quotient of $I \times S^1$ obtained by identifying the subspace $\{0\} \times S^1$ to a single point. Show that D is homeomorphic to the 2-disk D^2.

(c) Let X be a topological space, and let $f : I \times S^1 \to X$ be a map with the property that $f(\{0\} \times S^1)$ is a single point x_0. Show that the map f factors through the 2-disk D^2, in the sense that there exists a map $\overline{f}$ making the following diagram commute.

$$\begin{array}{ccc} I \times S^1 & \xrightarrow{f} & X \\ \downarrow & \nearrow_{\exists\, \overline{f}} & \\ D^2 & & \end{array}$$

Remark: We can use this exercise to show that, if γ is a loop in X that is trivial in $\pi_1(X)$, then we can 'fill' γ with a disk in the sense that there is a map $D^2 \to X$ whose restriction to $S^1 \subseteq D^2$ is γ. This remak does not need to make sense at this point.

8. **(Continuous, properly discontinuous group actions).** Let G be a group acting on a topological space X. This action is called *continuous*[1] if, for each $g \in G$, the map

$$\begin{aligned} g : X &\longrightarrow X \\ x &\longmapsto g \cdot x \end{aligned}$$

is continuous. The action is called *properly discontinuous* if each point $x \in X$ has some neighbourhood U_x with the property that $U_x \cap g(U_x) = \varnothing$ for every non-identity element $g \in G$.

Note: Despite the terminology, an action can be both continuous and properly discontinuous.

(a) Consider the action of $\mathbb{Z}$ on $\mathbb{R}$ by $n \cdot r = r + n$ for $n \in \mathbb{Z}$, $r \in \mathbb{R}$. Verify that this action is continuous and properly discontinuous.

(b) Find an example of a continuous group action of a group G on a topological space X that is free but not properly discontinuous.

(c) Let G_1 and G_2 be groups acting on spaces X_1 and X_2, respectively, continuously and properly discontinuously. Prove or find a counterexample: the action of $G_1 \times G_2$ on $X_1 \times X_2$ is continuous and properly discontinuous.

(d) Let X/G denote the quotient of X by the action of a group G. This means the quotient space (with the quotient topology) defined by the equivalence relation $x \sim g \cdot x$ for all $x \in X, g \in G$. Suppose that G acts continuously and properly discontinuously. Let $q : X \to X/G$ denote the quotient map. Show that quotient map has q has the property that, for each $\overline{x} \in X/G$, there is some neighbourhood $\overline{U}$ of $\overline{x}$ so that $q^{-1}(\overline{U})$ is a union of disjoint open sets in X, each of which is mapped homeomorphically by q to $\overline{U}$. We say q is a *covering map*.

9. **(Real projective space).** Define *real projective space* $\mathbb{RP}^n$ to be the quotient space (with the quotient topology) of $\mathbb{R}^{n+1} \setminus \{0\}$ by the equivalence relation $x \sim \lambda x$ for all $x \in \mathbb{R}^{n+1} \setminus \{0\}$ and $\lambda \in \mathbb{R} \setminus \{0\}$. The equivalence class of a point $x = (x_1, x_2, \ldots, x_n, x_{n+1})$ is denoted $[x_1 : x_2 : \cdots : x_n : x_{n+1}]$.

(a) Prove that $\mathbb{RP}^n$ is compact.

(b) Prove that $\mathbb{RP}^n$ is an n-dimensional manifold.
Hint: Write $\mathbb{RP}^n$ as a quotient of the orthogonal group $\mathrm{O}(n+1)$.
Alternate Hint: You may construct an atlas of charts explicitly. Consider the sets

$$U_i = \{[x_1 : x_2 : \cdots : x_{n+1}] \mid x_i \neq 0\} \subseteq \mathbb{RP}^n.$$

(c) **(Bonus)** For what values of n is $\mathbb{RP}^n$ orientable?

[1] We would need to modify this definition if G itself had topology, but in this case we assume G is discrete.