Notation

- $I = [0,1]$ (closed unit interval)
- $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ (closed unit n-disk)
- $S^n = \partial D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ (n-sphere) (we sometimes view S^1 as the unit circle in $\mathbb{C}$)
- $S^\infty = \bigcup_{n\geq 1} S^n$ with the weak topology
- Σ_g closed genus-g surface
- $\mathbb{R}\mathrm{P}^n$ real projective n-space
- $\mathbb{C}\mathrm{P}^n$ real complex n-space

Practice problems

1. Let X be a set with the trivial topology $\{X, \varnothing\}$. Show that X is contractible.
2. *Sierpiński space* is the space $S = \{0,1\}$ with the topology $\{\varnothing, \{1\}, \{0,1\}\}$. Show that S is contractible.
3. Classify the capital letters of the alphabet by homotopy type.
4. Suppose that spaces X, X' are homotopy equivalent, and Y, Y' are homotopy equivalent. Prove or disprove: it follows that $X \times Y$ is homotopy equivalent to $X' \times Y'$.
5. Let T be the torus $T = S^1 \times S^1$. Suppose we have maps
$$f : S^1 \vee S^1 \longrightarrow T \qquad \text{and} \qquad g : T \longrightarrow S^1 \vee S^1$$
Answer, with proof, the following questions. Is it possible for $f \circ g$ to be homotopic to the identity? Is it possible for $g \circ f$ to be homotopic to the identity?
6. Let $f, g : X \to Y$ be continuous maps of spaces.
   (a) Suppose X is contractible. Must f and g be homotopic?
   (b) Suppose Y is contractible. Must f and g be homotopic?
   (c) Suppose X is a wedge of circles and Y is simply connected. Must f and g be homotopic?
   (d) Suppose Y is a wedge of circles and X is simply connected. Must f and g be homotopic?
7. Let $D^2 = \{x \in \mathbb{R}^2 \mid |x| \leq 1\}$. Let $A \subseteq D^2$ be a subset of the disk that contains the boundary circle S^1 but does not contain 0. Prove that $\pi_1(A)$ contains a subgroup isomorphic to $\mathbb{Z}$.
8. Let X be a connected graph. Construct maps $f, g : X \to X$ so that $f \circ g = id_X$, but f and g do not induce isomorphisms on π_1.
9. A topological space G with a group structure is called a *topological group* if the group multiplication map and inverse map
$$\begin{aligned} G \times G &\longrightarrow G \\ (g,h) &\longmapsto g \cdot h \end{aligned} \qquad\qquad \begin{aligned} G &\longrightarrow G \\ g &\longmapsto g^{-1} \end{aligned}$$
are continuous. Examples include $(\mathbb{R}^n, +)$, $(\mathbb{C}, +)$, $(\mathbb{C}, \times)$, $(\mathbb{C} \setminus \{0\}, \times)$, $(S^1, \times)$, and products of these groups (like the torus).
   (a) Let $(G, \star)$ be a topological group with identity element e. Let α and β be loops based at e. Consider the loop $(\alpha \star \beta)(t)$ defined by taking the pointwise product of loops:
$$(\alpha \star \beta)(t) = \alpha(t) \star \beta(t).$$
Let $\cdot$ denote the product on $\pi_1(G, e)$. Show that $(\alpha \star \beta)$ is homotopic (rel $\{0,1\}$) to the product $\alpha \cdot \beta$.

(b) Show that $(\alpha \star \beta)$ is also homotopic (rel $\{0,1\}$) to the product $\beta \cdot \alpha$. Conclude that the fundamental group of a path-connected topological group must be abelian.

10. Recall that the suspension of a space X is the quotient of $X \times [0,1]$ obtained by collapsing $X \times \{0\}$ to a point and $X \times \{1\}$ to another point. Show that, if X is connected, then its suspension is simply connected.

11. Prove that the free group F_2 on 2 generators contains a copy of the free group F_n on n generators for every $n \geq 2$.

12. Let $p : \tilde{X} \to X$ be a covering map. Prove that, if $\tilde{X}$ is compact, then the cover must be finite-sheeted.

13. Let X be a CW complex. Use homotopy extension property to explain why any continuous map $f : X \to X$ is homotopic to a map with a fixed point.

14. Give an example (with proof) of a space X and a subspace A such that $H_n(X, A)$ is not isomorphic to $H_n(X/A)$.

15. Let X be a space and $x \in X$. Is $(X, X \setminus \{x\})$ ever a good pair?

16. Let X be a topological space with a finite number of path components $X_1, \ldots, X_N$. Use Mayer–Vietoris and induction to give a new calculation of the homology of X in terms of the homology of the spaces X_i.

17. Let $\widetilde{H}_n(X)$ denote the reduced homology of a space X in degree n. Verify that

$$\begin{aligned} \widetilde{H}_n : \underline{\text{Top}} &\longrightarrow \underline{\text{Ab}} \\ X &\longmapsto \widetilde{H}_n(X) \\ [f : X \to Y] &\longmapsto [f_* : \widetilde{H}_n(X) \to \widetilde{H}_n(Y)] \end{aligned}$$

is a covariant functor.

18. Consider $\mathbb{Q}$ as a subspace of $\mathbb{R}$. What can you say about the relative homology groups $H_n(\mathbb{R}, \mathbb{Q})$?

19. Define a chain homotopy, and prove that chain homotopic maps induce the same map on homology.

20. Let M be a Mobius band and let S be its boundary circle. Compute the homology of the quotient M/S using the long exact sequence of a pair. Verify your solution by a direct analysis of the homotopy type of the topological space M/S.

21. Construct a connected CW complex X such that $H_1(X) = 0$ but $\pi_1(X) \neq 0$.

22. For $n \geq 1$, consider the map $D^n \to S^n$ defined by collapsing ∂D^n. Does this map admit a section?

23. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$, and let

$$S^{n-1} = \{(x_1, x_2, \ldots, x_n, x_{n+1}) \in S^n \mid x_{n+1} = 0\}$$

be its equator. Prove or disprove: S^n retracts onto its equator.

24. Let X be the quotient of the 2-sphere $X = S^2/\{a, b\}$ gluing together the two points a and b. Let p be the image of $\{a, b\}$ in X.

    (a) Compute the local homology groups $H_2(X, X \setminus \{p\})$ and $H_2(X, X \setminus \{x\})$ for $x \neq p$.

    (b) Prove that any homeomorphism of X must fix the point p.

25. Let $X \subseteq K$ be a retract of K by a retraction $r : K \to X$. Show that $r_* : H_n(K) \to H_n(X)$ is a projection onto a direct summand.

26. (a) Let $A \subseteq X$. Prove or find a counterexample: for each n,

$$H_n(X) \cong H_n(A) \oplus H_n(X, A).$$

(b) Let $A \subseteq X$ and suppose A is a retract of X. Prove or find a counterexample: for each n,

$$H_n(X) \cong H_n(A) \oplus H_n(X, A).$$

27. Prove or disprove the analogue of the hairy ball theorem for the torus.

28. (a) Prove that punctured $\mathbb{R}\mathrm{P}^n$ is homotopy equivalent to $\mathbb{R}\mathrm{P}^{n-1}$.

(b) Use Mayer-Vietoris to give a new calculation of the homology of a non-orientable genus-g surface $N_g = \#_g \mathbb{R}\mathrm{P}^2$.

29. Compute the homology of $S^2 \times S^3$.

30. Must the following maps be nullhomotopic? Give a proof or prove a counterexample.

(a) $f : S^2 \to S^1 \times S^1$

(b) $g : S^1 \times S^1 \to S^2$

31. Let X be the union of the n-sphere S^n in $\mathbb{R}^{n+1}$ with the line segment of the x_{n+1}-axis connecting the north pole $(0, 0, \dots, 0, 1)$ to the south pole $(0, 0, \dots, 0, -1)$. Compute the fundamental group and homology groups of X.

32. Let X be the space obtained from a torus and a Mobius strip by gluing the boundary circle of the Mobius band to the meridian circle of the torus. Compute the fundamental group and homology of X.

33. Let T be a smoothly embedded torus in $\mathbb{R}^3$, as in Figure 1. Compute the homology of the quotient space $\mathbb{R}^3/T$.

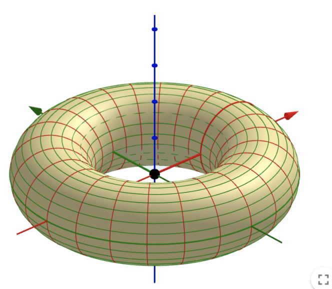

Figure 1: A torus embedded in $\mathbb{R}^3$.

34. Find a CW complex structure on the 3-torus $S^1 \times S^1 \times S^1$, and use it to compute its homology.

35. A Hausdorff space X is a *homology manifold of dimension* n if for every $x \in X$,

$$H_k(X, X \setminus \{x\}) = \begin{cases} \mathbb{Z}, & k = n \\ 0, & \text{otherwise} \end{cases}$$

Show that an n-dimensional manifold M is a homology manifold of dimension n.

36. For a space X, let $\widetilde{C}_*(X)$ denote the augmented singular chain complex of X (so $\widetilde{C}_*(X)$ agrees with $C_*(X)$ in every degree except for (-1)). Explain how to define chain maps ϕ, ψ in order to construct a short exact sequence of chain complexes

$$0 \longrightarrow \widetilde{C}_n(A \cap B) \xrightarrow{\phi} \widetilde{C}_n(A) \oplus \widetilde{C}_n(B) \xrightarrow{\psi} \widetilde{C}_n(A+B) \longrightarrow 0.$$

Use this result to prove a reduced version of the Mayer–Vietoris sequence

$$\cdots \longrightarrow \widetilde{H}_n(A \cap B) \xrightarrow{\Phi} \widetilde{H}_n(A) \oplus \widetilde{H}_n(B) \xrightarrow{\Psi} \widetilde{H}_n(X) \xrightarrow{\delta} \widetilde{H}_{n-1}(A \cap B) \longrightarrow \cdots$$

$$\cdots \longrightarrow \widetilde{H}_0(X) \longrightarrow 0.$$

You do not need to prove that the inclusion $\widetilde{C}_*(A+B) \subseteq \widetilde{C}_*(X)$ induces isomorphisms on homology.

37. The Hawaiian earring E is a compact space such that $H_1(E)$ is not finitely generated (in fact, it is not even countably generated). Prove, in contrast, that if X is a compact CW complex, its homology $\bigoplus_k H_k(X)$ is finitely generated.

38. Let Σ_g be a closed orientable surface of genus g. For which g is there a nontrivial covering map $p : \Sigma_g \to \Sigma_g$? Prove your answer.

39. Let $\mathbb{F}$ be a field, and let X be a finite CW complex. Show that

$$\chi(X) = \sum_i (-1)^i \dim_{\mathbb{F}} H_i(X, \mathbb{F}).$$

40. Let X and Y be finite CW complexes. Derive a formula for $\chi(X \vee Y)$ in terms of $\chi(X)$ and $\chi(Y)$.

41. Let N_g denote a non-orientable surface of genus g. Compute the homology of N_g with coefficients $\mathbb{Z}/2\mathbb{Z}$.

42. Let S^n denote a sphere of dimension $n \geq 0$. Let G be an abelian group. Compute the homology groups $H_i(S^n; G)$ ...
    (a) ... using the long exact sequence of a pair.
    (b) ... using Mayer–Vietoris.

    You may use without proof that the natural analogues of both these long exact sequences hold for homology with coefficients.

43. Let P_n be a regular $(2n)$-gon with parallel edges identified by a translation. Classify the surface P_n.

44. Let X be the surface constructed from Σ_g by deleting an open disk and then gluing in a Mobius band along its boundary. Classify the surface X.

45. **True or counterexample.** For each of the following statements: if the statement is true, write "True". If not, state a counterexample. No justification necessary.
    Note: If the statement is false, you can receive partial credit for writing "False" without a counterexample.
    (i) Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps of spaces. If one of f or g is nullhomotopic, then $g \circ f$ is nullhomotopic.
    (ii) Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps of spaces. If $g \circ f$ is nullhomotopic, then one of f or g must be nullhomotopic.
    (iii) Let $F_t : X \to X$ be a homotopy of maps $F_0, F_1 : X \to X$. Let $A \subseteq X$. If $F_t(A) \subseteq A$ for all t, then F_t induces a well-defined homotopy of functions $X/A \to X/A$.
    (iv) If A is deformation retract of X, then A and X are homeomorphic.
    (v) Let X be a finite CW complex. If X is simply connected, then X is contractible.
    (vi) Every map from $\mathbb{R}\mathrm{P}^1$ to $\mathbb{C}\mathrm{P}^1$ is nullhomotopic.

(vii) Let U, V be open subsets of a space X. If U and V are path-connected, and $U \cup V$ is simply connected, then $U \cap V$ is connected.

(viii) Let $p : \tilde{X} \to X$ be a covering map. Then $p_* : H_n(\tilde{X}) \to H_n(X)$ is injective for every n.

(ix) Let $B \subseteq A \subseteq X$. If A is a deformation retract of B, then $H_n(X, A) \cong H_n(X, B)$ for all n.

(x) If $f : X \to Y$ is a homotopy equivalence, then f must surject

(xi) If $f : S^n \to S^n$ is a homotopy equivalence, then f must surject

(xii) If a map of spheres $f : S^n \to S^n$ has no fixed points, then f has degree $(-1)^{n+1}$

(xiii) For any $n \geq 1$, there exists a map $f : S^n \to S^n$ of every degree $d \in \mathbb{Z}$.

(xiv) If a continuous map $f : S^n \to S^n$ is a local homeomorphism at a point $x \in S^n$, then the local degree of f at x must be ± 1.

(xv) Let $n \in \mathbb{Z}_{\geq 1}$. If $p : S^n \to X$ is a finite-sheeted cover, then it must be 1- or 2-sheeted.

(xvi) If the antipodal map $S^n \to S^n$ is homotopic to the identity map, then n must be odd.

(xvii) If a space X is simply connected, then $H_0(X) = H_1(X) = 0$.

(xviii) If $H_0(X) = H_1(X) = 0$, then X is simply connected.

(xix) Let X be a space and A a subspace. If A is contractible, then $H_n(X, A) = \tilde{H}_n(X)$ for all n.

(xx) Let X be a space and A a subspace. If A is contractible, then $\tilde{H}_n(X/A) = \tilde{H}_n(X)$ for all n.

(xxi) If X is a CW complex of dimension n, then $H_{n+1}(X) = 0$.

(xxii) If X is a CW complex of dimension n, then $H_n(X) \neq 0$.

(xxiii) If X is a CW complex of dimension n, then $H_n(X)$ is free abelian.

(xxiv) If a CW complex X has no cells of dimension d, then $H_d(X) = 0$.

(xxv) If a CW complex X has $H_d(X) = 0$, then X has no cells of dimension d.

(xxvi) If a CW complex has cells in only even dimensions, then $H_n(X) = C_n(X)$ for all n, where $C_n(X)$ denotes the cellular chains on X.

(xxvii) Let X be a CW complex with k-skeleton X^k. Then for all n, $H_n(X^n, X^{n-1})$ is free abelian.

(xxviii) Let X be a finite CW complex. If $\chi(X) = 1$, then X is contractible.

(xxix) Let X be a finite CW complex. Then every map $f : X \to X$ that is homotopic to the identity must have a fixed point.

(xxx) Let X be a contractible compact manifold. Then every continuous map $f : X \to X$ has a fixed point.

(xxxi) Let X be a contractible manifold. Then every continuous map $f : X \to X$ has a fixed point.

46. Explain the value in defining and studying the fundamental group $\pi_1(X)$ of a space X.

47. Explain the value in defining and studying the homology groups $H_*(X)$ of a space X.

48. Suppose that I am convinced that absolute homology groups $H_*(X)$ of a space X are a useful homotopy invariant, but I do not know why we define relative homology groups $H_*(X, A)$. Explain the value of defining relative homology groups as a tool to prove results about absolute homology groups.

49. Write a bullet-point summary of all the major results we have proved about ...
    (a) fundamental group
    (b) covering spaces
    (c) homology

50. What tools do we have to compute the following for a given space X? How might we recognize which tool to try?

(a) fundamental group
(b) covering spaces
(c) homology

51. Write a bullet-point summary of all the major results we have proved about the following spaces.

You may wish to include:

- their definition
- CW complex structures, Δ-complex structures,
- whether they are compact, connected, path-connected, locally path-connected, semi-locally simply-connected, contractible
- π_1
- their universal covers, other covers
- their homology, their homology with coefficients in $\mathbb{Z}/2\mathbb{Z}$
- their Euler characteristics
- any other results, such as fixed point theorems or results on vector fields

(a) $\mathbb{R}^n$
(b) disks D^n
(c) spheres S^n
(d) n-tori $(S^1)^n$
(e) graphs
(f) (orientable or nonorientable) closed surfaces
(g) punctured surfaces
(h) projective spaces $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$

52. (a) **(Topology Qual, May 2020).** Which of the following groups are fundamental groups of compact surfaces without boundary? For those which are, classify the surface:
(i) $\langle a, b, c | abca^{-1}b^{-1}c \rangle$
(ii) $\langle a, b, c, d | abcda^{-1}b^{-1}c^{-1}d^{-1} \rangle$
(iii) $\langle a, b, c | abcb^{-1}a^{-1}c \rangle$.

(b) **(Topology Qual, Sep 2018).** Consider two disjoint squares $ABCD$, $EFGH$ in $\mathbb{R}^2$. Identify their sides as follows:

$$AD \text{ with } HG,$$
$$DC \text{ with } EH,$$
$$AB \text{ with } BC,$$
$$EF \text{ with } FG.$$

All identifications of sides are bijective linear, with the endpoints identified in the order given. Is the quotient space of the identification a compact surface (i.e. a compact topological 2-manifold)? If so, classify it.

(c) **(Topology Qual, May 2018).** Let X be the space obtained by removing the open square in $\mathbb{R}^2$ with vertices $(11), (12), (21), (22)$ from the closed square with vertices $(00), (03), (30), (33)$. Now let X be the space obtained by identifying the following pairs of line segments, directions indicated, via affine bijective maps:

$$(00), (03) \text{ with } (21), (22),$$
$$(30), (33) \text{ with } (11), (12),$$
$$(00), (30) \text{ with } (22), (12),$$
$$(03), (33) \text{ with } (21), (11),$$

(i) Calculate $\pi_1(X)$.

(ii) Prove that X is a compact surface, and classify it.

(d) **(Topology Qual, Jan 2018).** Let Z be a convex 10-gon in the plane with vertices $A_0, A_1, A_2, A_3, A_4, B_4, B_3, B_2, B_1, B_0$ appearing in this order on the boundary (oriented counter-clockwise). Let X be the topological space obtained from Z by gluing the line segments A_0A_1 with B_2B_3, B_0B_1 with A_2A_3, A_1A_2 with B_1B_2, A_3A_4 with B_3B_4, A_0B_0 with B_4A_4. All pairs of line segments are attached by linear maps with the vertices corresponding in the order listed (first to first, last to last).

(i) Calculate $\pi_1(X)$.

(ii) Classify the surface X.

(e) **(Topology Qual, Jan 2017).**
Let $A_k = e^{2k\pi i/2n}$. Let C_n be the convex hull of $\{A_k \mid k = 0, 1, \dots, 2n-1\}$ with the topology induced from $\mathbb{C}$. Let $\sim$ be the smallest equivalence relation on C_n such that $tA_k + (1-t)A_{k+1} \sim (1-t)A_{k+n} + tA_{k+n+1}$, for all $k \in \mathbb{Z}/2n$, $0 \le t \le 1$. Let $X_n = C_n/\sim$ with the quotient topology.

(i) Calculate $\pi_1(X_n)$.

(ii) Classify the surface X_n

53. **(Topology Qual, Jan 2021).** Let G be a topological space admitting a topological group structure, i.e., one has a continuous multiplication map $\mu : G \times G \to G$ and a continuous inversion map $\iota : G \to G$ that define a group structure on the set G. Assume that G is homeomorphic to a connected finite CW complex. Show that $\chi(G) = 0$ unless $G = \{1\}$.

54. **(Topology Qual, Aug 2019).** Let a CW complex X be obtained from a k-sphere, $k \ge 1$, by attaching two $(k+1)$-cells along attaching maps of degrees $m, n \in \mathbb{Z}$. Calculate the homology of X.

55. **(Topology Qual, May 2019).** Let S^1 be the unit sphere in $\mathbb{C}$, let $T = S^1 \times S^1$ and let $T' = T/(S^1 \times \{1\})$. Let X be the connected sum of T and T', i.e. a space obtained by cutting out interiors of closed 2-disks from T and T', respectively, (disjoint from the singular point in case of T') and attaching the resulting spaces by the boundaries of the disks. Compute the fundamental group and homology of X.

56. **(Topology Qual, May 2019).** For which values of $g \ge 0$ is it true that for every number $h \ge g$ (g, h integers), a compact oriented surface X of genus g (without boundary) has a covering $f : Y \to X$ where Y is a compact oriented surface of genus h?

57. **(Topology Qual, Jan 2019).** Let S_1, S_2 be two disjoint copies of the n-sphere, $n > 1$ fixed. Choose two distinct points $A_i, B_i \in S_i$. Let Z be a space obtained from $S_1 \sqcup S_2$ by identifying $A_1 \sim A_2$, $B_1 \sim B_2$. Compute, with proof, the lowest possible number of cells in a CW decomposition of Z.

58. **(Topology Qual, May 2017).** Let S^1 be the set of complex numbers of absolute value 1 with the induced topology. K be the quotient space formed from $S^1 \times [0,1]$ by identifying every point $(z, 0)$ with the point $(z^{-2}, 1)$. Compute the homology of K.

59. **(Topology Qual, May 2017).** Let X be a connected CW-complex such that $H_i(X) = 0$ for all $i > 0$. Let S^k denote the k-sphere. Prove that for all $k \in \mathbb{N}$, $H_n(X \times S^k)$ is $\mathbb{Z}$ for $n = 0$ and $n = k$, and 0 for all other values of n.

60. **(Topology Qual, Sep 2016).** Let $Z = \{(x, y) \in \mathbb{C}^2 \mid x = 0 \text{ or } y = 0\}$. Find the homology of $\mathbb{C}^2 \setminus Z$ (with the subspace topology induced from the Euclidean topology on $\mathbb{C}^2$).

61. **(Topology Qual, Jan 2016).** Let $U, V \subseteq S^n$, $n \ge 2$, be two non-empty connected open subsets such that $S^n = U \cup V$. Show that $U \cap V$ is connected.

62. **(Topology Qual, Jan 2016).** Fix a prime number p. Let X be a finite CW complex with an action of $G = \mathbb{Z}/p$.

(a) If $\chi(X)$ is not divisible by p, show that the G action on X has a fixed point.

(b) Give an example of such an action that is fixed-point free with $\chi(X) = 0$.