Terms and concepts covered: Homotopy; homotopic maps; nullhomotopic map. Homotopy rel a subspace. Homotopy equivalence; homotopy type; contractible. Deformation retraction. CW complex; weak topology. Products, wedge sums, and quotients of CW complexes.

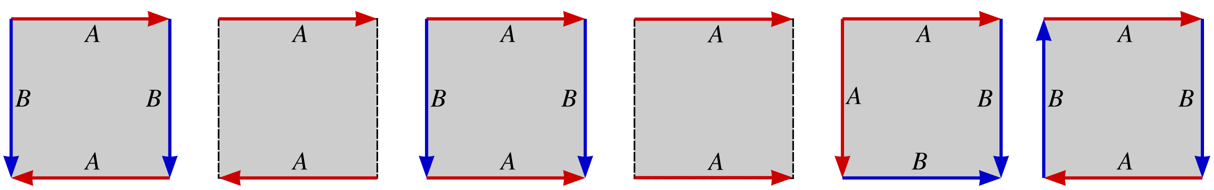
Corresponding reading: Hatcher, Chapter 0, "Homotopy and homotopy type", "Cell complexes", "Operations on spaces" & Hatcher, Appendix, through Prop A.3.

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let X be a topological space, and let $f, g : X \to \mathbb{R}^n$ be continuous maps. Show that f and g are homotopic via the homotopy
$$F_t(x) = t\, g(x) + (t-1) f(x).$$
2. Let X be a topological space. Show that all constant maps to X are homotopic if and only if X is path-connected. In general, what are the homotopy classes of constant maps in X?
3. Recall that a subset $S \subseteq \mathbb{R}^n$ is *star-shaped* if there is a point $x_0 \in S$ such that, for any $x \in S$, the line segment from x_0 to x is contained in S. Show that any star-shaped subset of $\mathbb{R}^n$ is contractible. Conclude in particular that convex subsets of $\mathbb{R}^n$ are contractible.
4. Let X be a space and let $A \subseteq X$ be a deformation retract. Verify that X and A are homotopy equivalent.
5. Let $S^1 \subseteq \mathbb{R}^2$ be the unit circle. Find the mistake in the following "proof" that S^1 is contractible.

   False proof. There is a deformation retraction from S^1 to the point $(1,0)$ given by the homotopy
$$F_t(x,y) = \frac{(1-t)(x,y) + t(1,0)}{||(1-t)(x,y) + t(1,0)||}.$$
6. Suppose X and Y are homotopy equivalent spaces. Show that Y is path-connected if and only if X is.
7. **(Quotient surfaces).** Identify among the following quotient spaces: a cylinder, a Möbius band, a sphere, a torus, real projective space, and a Klein bottle.



8. Describe (with pictures!) how to construct the following CW complex structures. No proofs necessary.
   (a) Describe a CW complex structure on a 1-sphere S^1, and use this structure to construct a CW complex structure on a torus $T = S^1 \times S^1$.
   (b) Describe a CW complex structure on a genus-2 surface.
   (c) Describe a CW complex structure on an n-sphere S^n that has 2 cells in each dimension.
   Hint: Work inductively.
9. Let $X = \bigcup_n X^n$ be a CW complex with n-skeleton X^n. Recall that we defined the topology on X so that a set U is open iff $U \cap X^n$ is open for every n.
   (a) Suppose that X is finite-dimensional, that is, $X = X^N$ for some N. Show that the topology on X agrees with the topology from our inductive definition of the N-skeleton X^N as a quotient space.

(b) Again let X be any CW complex. Show that a set $C \subseteq X$ is closed iff $C \cap X^n$ is closed for every n.

10. We define a *subcomplex* of a CW complex X to be a closed subset that is equal to a union of cells. Show that a subcomplex A is itself a CW complex, by verifying inductively that the images of the attaching map of an n-cell in A must be contained in its $(n-1)$-skeleton A^{n-1}.

11. Verify the details of the natural CW complex structure on a product of CW complexes, or a quotient of a CW complex by a subcomplex.

12. Prove that any finite CW complex is compact, by realizing it as the continuous image of a finite union of closed balls.

Assignment questions

(Hand these questions in!)

1. **(Homotopy defines an equivalence relation).**
   (a) Prove the following lemma.

   Lemma (Pasting Lemma). Let A, B be a topological spaces, and suppose A is the union $A = A_1 \cup A_2$ of closed subsets A_1 and A_2. Then a map $f : A \to B$ is continuous if and only if its restrictions $f|_{A_1}$ and $f|_{A_2}$ to A_1 and A_2, respectively, are continuous.

   (b) Let X, Y be topological spaces and consider the set of continuous maps $X \to Y$. Show that the relation "f is homotopic to g" defines an equivalence relation on this set.

2. **(Homotopy equivalence is an equivalence relation).** Show that "homotopy equivalence" defines an equivalence relation on topological spaces.

3. **(S^∞ is contractible).** Define the infinite-dimensional sphere S^∞ as the space

$$S^\infty = \bigcup_{n \geq 0} S^n = \left\{ (x_1, x_2, \dots,) \;\middle|\; x_i \in \mathbb{R},\; x_i = 0 \text{ for all but finitely many } i, \sum_i x_i^2 = 1 \right\}$$

   It is topologized so that a subset U is open if and only if $U \cap S^n$ is open for every n.
   Show that S^∞ is contractible. *Hints:*

   - The map $S^\infty \to S^\infty$ given by $(x_1, x_2, x_3, \dots) \longmapsto (0, x_1, x_2, x_3, \dots)$ is continuous.
   - Warm-up Problem 5.

4. **(A CW complex structure on the sphere).** Let S^n denote the n-sphere. In general we understand S^n is defined up to homeomorphism, but for the purposes of this question we will concretely define S^n to be the unit sphere in $\mathbb{R}^{n+1}$,

$$S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid |\mathbf{x}| = 1\}.$$

   Let D^n denote the closed n-ball. Again, to be concrete we take D^n to be the unit ball in R^n,

$$D^n = \{\mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x}| \leq 1\}.$$

   (a) Prove the following theorem.

   Theorem (A homeomorphism criterion). Let $f : X \to Y$ be a continuous, bijective map of topological spaces. Suppose X is compact and Y is Hausdorff. Then f is a homeomorphism.

   (b) Let X be a topological space, and let $\sim$ be an equivalence relation on X. Let $X/\sim$ denote the corresponding quotient space, and $q : X \to X/\sim$ the quotient map. State the definition of the quotient topology on $X/\sim$, and state the universal property of the quotient map q.

(c) Let $D^n/\sim$ be the quotient of D^n obtained by identifying all points in the boundary to a single point. Use parts (a) and (b) to prove that $D^n/\sim$ is homeomorphic to S^n.

Remark: Going forward, you may assert without proof the identity of quotient spaces such as this one and the ones in Warm-Up Problems 7 and 8. But we should check this rigorously at least this once!

5. **(CW complexes are Hausdorff).**

Definition (The characteristic map). Let X be a CW complex. For each n-cell e^n_α the associated *characteristic map* Φ_α is the composition

$$\Phi_\alpha : D^n_\alpha \hookrightarrow X^{n-1} \bigsqcup_\beta D^n_\beta \longrightarrow X^n \longrightarrow X$$

Specifically $\Phi_\alpha|_{\partial D^n}$ is the attaching map ϕ_α, and Φ_α maps the interior of D^n homeomorphically to e^n_α.

(a) Let X be a topological space. Recall that topologists say "points are closed" in X to mean that the singleton set $\{x\}$ is closed for all $x \in X$. Hatcher writes (p522), "Points are closed in a CW complex X since they pull back to closed sets under all characteristic maps Φ_α." Explain why points pull back to closed sets, and explain why this observation implies that points are closed.

(b) Prove that a CW complex is Hausdorff. *Hint:* Read the first half of Hatcher p522, and explain the proof of Proposition A.3 in the special case that A and B are points. You may use the book as a reference while you write this proof, though you should not simply copy the book!

6. **(Compact subsets of CW complexes and the closure-finite property).**

(a) Let X be a CW complex, and let S be a (possibly infinite) subset of X such that every point of S is in a distinct cell of X. Prove that S is closed. Since the same argument applies to any subset of S, conclude that S has the discrete topology.

(b) Prove the following lemma.

Lemma (Compact subsets of CW complexes). Let X be a CW complex. Any compact subset of X intersects only finitely many cells.

(c) Show that the closure of e^n_α in X is equal to the image of the characteristic map Φ_α (Question 5).

(d) "CW" stands for "closure-finiteness, weak topology". Prove the "closure-finiteness" property.

Proposition (Closure-finiteness of CW complexes). Let X be a CW complex. The closure of any cell intersects only finitely many other cells.

(e) The *Hawaiian earring* is a subspace of $\mathbb{R}^2$ defined as the union $\bigcup_{n\geq 1} C_n$, where C_n is the circle of radius $\frac{1}{n}$ and center $(\frac{1}{n}, 0)$. See Figure 2. It is a favourite source of counterexamples in algebraic topology.

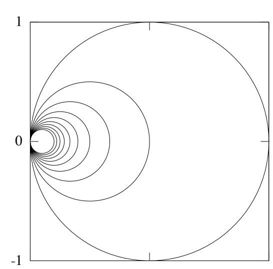


Figure 2: The Hawaiian Earring

Show that the topology on the Hawaiian earring does not agree with the weak topology on a countable wedge of circles.

Remark: In fact, the Hawaiian earring is not even homotopy equivalent to a CW complex.

7. **(Bonus: Homotopies as paths of maps).** Let X and Y be locally compact, Hausdorff topological spaces. Consider the space $C(X,Y)$ of continuous maps from X to Y with the compact-open topology. Let I be a closed interval. Show that the definition of a homotopy of maps $X \to Y$ is equivalent to the definition of a continuous map $I \to C(X,Y)$.

Wellbeing

(This section is completely optional. This is a nudge to prioritize your wellbeing during the pandemic.)

1. Choose some concrete health goals for the semester, especially concerning sleep, exercise, and/or nutrition. Some standard tips:
   - Make your goals SMART: specific, measurable, achievable, relevant, and time-bound.
     For example, "walk for at least 40 minutes, at least 5 times per week" is a better goal than "get more exercise".
   - If you wish to make changes compared with last term, then keep your goals realistic by beginning with an honest assessment of your old habits, and decide on some initial small steps to improve on them.
   - Choose one or two goals to be your priorities.
   - Break your goals into smaller steps. Find small ways to reward yourself for success.
   - "Gamify" your goals by tracking your progress. For example, use a step-counter, or set up a calendar where you can toggle a button for each daily goal you have met.
   - Anticipate some of the obstacles you will face. What can you do, specifically, to mitigate them?
   - Find ways to create accountability for yourself, such as an "accountability buddy".
   - Be prepared for setbacks. Decide in advance what to do on days you do not meet your goals: forgive yourself, modify the goals if appropriate, and re-commit.