Terms and concepts covered: Five lemma, singular $\cong$ simplicial homology, degree of a map $S^n \to S^n$, properties of degree, local degree and its relationship to degree, cellular homology.

Corresponding reading: Hatcher Ch 2.1, "The Equivalence of Simplicial and Singular Homology", Ch 2.2, "Degree", "Cellular Homology" (up to Example 2.36), "Mayer–Vietoris Sequences" (up to / including Example 2.46) .

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. (a) Show that S^n has a Δ-complex structure defined inductively by gluing together two n-simplices Δ_1^n and Δ_2^n by the identity map on their boundaries.
   (b) Show that (with suitably chosen orientations) the corresponding simplicial homology group $H_n(S^n) \cong \mathbb{Z}$ is generated by the cycle $\Delta_1^n - \Delta_2^n$.
   (c) Compute the map induced on $H_n(S^n)$ by the reflection that fixes the equator S^{n-1} and interchanges the two simplices.
2. Suppose that $f : S^n \to S^n$ has no fixed points. Show that

$$f_t(x) = \frac{(1-t)f(x) - tx}{||(1-t)f(x) - tx||}$$

   is a homotopy from f to the antipodal map $x \mapsto -x$. (Why does this homotopy require no fixed points?)
3. (a) Let $f : S^n \to S^n$ be a homeomorphism. Show that deg(f) must be ± 1.
   (b) Suppose that a continuous map $f : S^n \to S^n$ is not surjective, so f factors through a map

$$S^n \to S^n \setminus \{x\} \hookrightarrow S^n.$$

   Show that $\deg(f) = 0$.
   (c) Show that, if $f \simeq g$, then $\deg(f) = \deg(g)$.
   (d) If $f, g : S^n \to S^n$, show that $\deg(f \circ g) = \deg(g)\deg(f)$.
   (e) Let $f : S^n \to S^n$ be a homotopy equivalence. Show that $\deg(f)$ must be ± 1.
   (f) Show that a reflection $S^n \to S^n$ has degree -1.
   (g) Show that the antipodal map $x \mapsto -x$ is the product of $(n+1)$ reflections. Conclude that it has degree $(-1)^{n+1}$.
4. Suppose that a continuous map $f : S^n \to S^n$ is not surjective. Show that f is nullhomotopic.
5. (a) Explain why any map $S^n \to S^n$ that factors $S^n \to D^n \to S^n$ must be degree zero.
   (b) Construct a surjective map $S^n \to S^n$ of degree zero.
6. Let $n \geq 1$. Use the Homotopy Extension Property to explain why every map $S^n \to S^n$ can be homotoped to have a fixed point.
7. Let $x \in S^n$.
   (a) Describe a generator of $H_n(S^n, S^n \setminus \{x\})$.
   (b) Show that $H_n(S^n, S^n \setminus \{x\}) \cong H_n(U, U \setminus \{x\})$ for any neighbourhood U of x.

(c) Let $f : S^n \to S^n$ be a continuous map. Let y be a point with a finite preimage $f^{-1}(y) = \{x_1, \dots, x_m\}$. Let $U_1, \dots U_m$ be small disjoint open balls around $x_1, x_2, \dots, x_m$, respectively, that map to a small open ball V about y. Show that we can compute the local degree

$$f_* : H_n(U_i, U_i \setminus \{x_i\}) \longrightarrow H_n(V, V \setminus \{y\})$$

by computing the degree

$$f_* : H_{n-1}(U_i \setminus \{x_i\}) \longrightarrow H_{n-1}(V \setminus \{y\})$$

and give a topological description of the latter map.

8. Let SX denote the suspension of a space X (Assignment Problem 4 (e)). Convince yourself that there is a homeomorphism $SS^n \cong S^{n+1}$ for all $n \geq 0$.

Assignment questions

(Hand these questions in!)

1. (a) Recall that a *tangent vector field* to the unit sphere $S^n \subseteq \mathbb{R}^{n+1}$ is a continuous map $v : S^n \to \mathbb{R}^{n+1}$ such that $v(x)$ is tangent to S^n at x, i.e., $v(x)$ is perpendicular to the vector x for each x. Let $v(x)$ be a nonvanishing tangent vector field on the sphere S^n. Show that

$$f_t(x) = \cos(\pi t)x + \sin(\pi t)\left(\frac{v(x)}{||v(x)||}\right)$$

is a homotopy from the identity map $id_{S_n} : S^n \to S^n$ to the antipodal map $-id_{S_n} : S^n \to S^n$.

(b) Prove the following theorem.

Theorem (Hairy ball theorem). The sphere S^n admits a nonvanishing continuous tangent vector field if and only if n is odd.

Remark: This result is alternately called the "Hedgehog Theorem" or the "'You can't comb a coconut' Theorem".

2. **(Topology Qual, Sep 2017).** Prove that for positive integers n, k, there does not exist a covering $\pi : S^{2n} \to X$ where X is a simplicial complex with $\pi_1(X) \cong \mathbb{Z}/(2k+1)$.

3. (a) Consider the map

$$\begin{aligned} f : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto z^k \end{aligned}$$

Compute $f_* : H_2(\mathbb{C}, \mathbb{C} \setminus \{0\}) \to H_2(\mathbb{C}, \mathbb{C} \setminus \{0\})$.

(b) Let $f : \mathbb{C} \to \mathbb{C}$ be a degree-d polynomial with complex coefficients. The function f then extends to a function

$$\widehat{f} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$$

on the Riemann sphere $\widehat{\mathbb{C}} \cong S^2$, the one-point compactification of $\mathbb{C}$. Prove that the degree of the map $\widehat{f}$, as a map $S^2 \to S^2$, is d.

4. **(Mayer–Vietoris).**

(a) Let X be a space, and let $A, B \subseteq X$ be subspaces whose interiors cover X. Let $C_n(A+B)$ denote the subgroup of the singular n-chain group $C_n(X)$ consisting of chains that are sums of a chain in A and a chain in B. Show that following is a short exact sequence of chain complexes.

$$0 \longrightarrow C_n(A\cap B) \xrightarrow{\phi} C_n(A)\oplus C_n(B) \xrightarrow{\psi} C_n(A+B) \longrightarrow 0$$

$$x \longmapsto (x,-x)$$

$$(y,z) \longmapsto y+z$$

(b) We will not prove this carefully, but it is possible to show (by subdividing simplices) that the inclusion of chain complexes

$$C_*(A+B) \to C_*(X)$$

induces isomorphisms on homology groups. Use this fact to deduce the following theorem, and describe the maps Φ and Ψ.

Theorem (The Mayer–Vietoris long exact sequence). Let X be a space, and let $A, B \subseteq X$ be subspaces whose interiors cover X. Then there is a long exact sequence on homology groups

$$\cdots \longrightarrow H_n(A\cap B) \xrightarrow{\Phi} H_n(A)\oplus H_n(B) \xrightarrow{\Psi} H_n(X) \xrightarrow{\delta} H_{n-1}(A\cap B) \longrightarrow \cdots$$

$$\cdots \longrightarrow H_0(X) \longrightarrow 0.$$

(c) Verify the following statement from Hatcher (p150) about the connecting homomorphism δ. You do not need to verify the claim about barycentric subdivision.

"The boundary map $\delta : H_n(X) \to H_{n-1}(A\cap B)$ can easily be made explicit. A class $\alpha \in H_n(X)$ is represented by a cycle z, and by barycentric subdivision or some other method we can choose z to be a sum $x+y$ of chains in A and B, respectively. It need not be true that x and y are cycles individually, but $\partial x = -\partial y$ since $\partial(x+y)=0$, and the element $\delta\alpha \in H_{n-1}(A\cap B)$ is represented by the cycle $\partial x = -\partial y$, as is clear from the definition of the boundary map in the long exact sequence of homology groups associated to a short exact sequence of chain complexes."

(d) Use the Mayer–Vietoris sequence fo inductively re-compute the homology of S^n. *Hint:* Take A to be a neighbourhood of the top hemisphere, and B a neighbourhood of the bottom hemisphere.

(e) **Definition (Suspension).** For a topological space X, the *(unreduced) suspension* SX of X is the quotient of $X \times I$ obtained by collapsing $X \times \{0\}$ to one point and collapsing $X\times\{1\}$ to another point.

Use the Mayer–Vietoris long exact sequence to prove that $\widetilde{H}_n(SX) \cong \widetilde{H}_{n-1}(X)$.
Hint: First explain why the images of $X \times [0, 0.6)$ and $X \times (0.4, 1]$ in SX are contractible.

(f) **(Topology Qual, Jan 2020).** The *unreduced suspension* $\tilde{X}$ of a space X is obtained from $X \times [0,1]$ by identifying $(x,0)\sim(y,0)$ and $(x,1)\sim(y,1)$ for all choices of points $x, y \in X$. If S^n is the n-sphere, $n>0$, compute the homology of the unreduced suspension of $S^n \times \{0,\ldots,k\}$.

5. Recall our CW structures on $\mathbb{RP}^n$ and $\mathbb{CP}^n$ from Homework 4 Problem #3.
   (a) Compute the cellular homology of $\mathbb{CP}^n$.
   (b) **(Topology Qual, Jan 2021).** Let $\pi : \mathbb{C}^3\setminus\{0\} \to \mathbb{CP}^2$ be the natural map, sending a point $x \in \mathbb{C}^3\setminus\{0\}$ to the line $\ell_x \in \mathbb{CP}^2$ connecting x to 0 in $\mathbb{C}^3$. Does π admit a section (i.e., a right-inverse)?
   (c) Compute the cellular homology of $\mathbb{RP}^n$.
   (d) **(Topology Qual, Jan 2018).** Prove that every CW-structure on $\mathbb{RP}^n$ has at least one cell in each dimension $0,1,\ldots,n$.
   (e) **(Topology Qual, Aug 2020).** Let $f: S^4 \to S^4$ be a map with the property that $f(x)=f(y)$ if y is the antipode of x. Show that $H_4(f)=0$.

6. **(Topology Qual, May 2019).** Let X be a 2-dimensional CW-complex with one 0-cell, four 1-cells a,b,c,d and four 2-cells, attached along the loops $a^2bc, ab^2d, ac^2d, bcd^2$. Compute the homology of X.

Wellbeing

(This section is completely optional. This is a nudge to prioritize your wellbeing during the pandemic.)

1. **(Health comes first).** This week, re-evaluate and update your health and wellbeing goals. What can you do to look after yourself in the last four weeks of the term?