Terms and concepts covered: local degree and its relationship to degree, cellular homology, Eilenberg–Steenrod axioms for a homology theory.

Corresponding reading: Hatcher Ch 2.2, "Degree", "Cellular Homology", "Mayer–Vietoris Sequences", "Homology with coefficients", Ch 2.3 "Axioms for homology".

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Verify the claim on the quiz about the CW structure on the nonorientable surface
$$N_g = \mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2 \# \cdots \# \mathbb{R}\mathrm{P}^2.$$
2. We outlined proofs of the following facts about the homology of a CW complex X. Verify the facts directly in the case that the CW complex stuctue on X is a Δ-complex structure, by considering the simplicial homology groups.
   (a) If X is finite dimensional, $H_k(X) = 0$ for all $k > \dim(X)$.
   (b) More generally, for any Δ-complex X, $H_k(X^n) = 0$ for all $k > n$.
   (c) The inclusion $X^n \hookrightarrow X$ induces isomorrphisms $H_k(X^n) \xrightarrow{\cong} H_k(X)$ for all $k < n$.
   (d) The inclusion $X^n \hookrightarrow X$ induces a surjection $H_n(X^n) \twoheadrightarrow H_n(X)$.
3. Let X be a CW complex. Prove that the path-components of X are the path-components of its 1-skeleton X^1. Conclude that the map
$$H_0(X^k) \to H_0(X)$$
induced by the inclusion of the k-skeleton is an isomorphism for all $k \geq 1$.
4. Prove that a homeomorphism of spaces $X \to Y$ gives rise to an isomorphism of singular chain complexes $C_*(X) \cong C_*(Y)$. Give conditions for a map of pairs $(X, A) \to (Y, B)$ give rise to an isomorphism of chain complexes.
5. Verify that the Δ-complex structures we used on the torus and $\mathbb{R}\mathrm{P}^2$ are not triangulations. Can you further subdivide the simplices to triangulate these spaces?

Assignment questions

(Hand these questions in!)

1. Use the Mayer–Vietoris sequence to re-compute the homology of the genus-g surface Σ_g, using the definition of Σ_g as the connected sum $\Sigma_{g-1} \# \Sigma_1$.
Hint: First explain why a punctured genus-h surface is homotopy equivalent to a wedge of $2h$ circles.
2. **Definition (Euler characteristic).** Let X be a finite CW complex. The *Euler characteristic* $\chi(X)$ of X is defined to by the alternating sum
$$\chi(X) = \sum_i (-1)^n c_n,$$
where c_n is the number of n-cells of X.
   (a) Prove the following theorem. This is a purely algebraic, general result about chain complexes. Recall that the *rank* of an abelian group is the cardinality of a maximal linearly independent subset.

Theorem (Euler characteristic via homology). Let X be a finite CW complex with Euler characteristic $\chi(X)$. Then

$$\chi(X) = \sum_i (-1)^n \mathrm{rank}(H_n(X)).$$

Notably, this shows that $\chi(X)$ only depends on the topology of X (in fact, only the homotopy type of X), and not on a particular choice of CW complex structure.

(b) Suppose that X and Y are finite CW complexes. Prove that $\chi(X \times Y) = \chi(X)\chi(Y)$.

(c) State the Euler characteristic of each of the following spaces. No justifcation needed.

- Euclidean space $\mathbb{R}^n$ or disk D^n
- Tree T
- Wedge of circles $\bigvee_n S^1$
- Sphere S^n
- n-torus $(S^1)^n$
- Real projective space $\mathbb{R}\mathrm{P}^n$
- Complex projective space $\mathbb{C}\mathrm{P}^n$
- Orientable closed genus-g surface Σ_g
- Nonorientable closed genus-g surface N_g

(d) Given a finite-sheeted cover $p : \tilde{X} \to X$ of a finite CW complex X, describe how to construct a CW complex structure on $\tilde{X}$. You do not need to check point-set details.
Hint: Compare to Homework 6 Problem 3(a).
Use your construction to prove the following theorem.

Theorem (Euler characteristic of a cover). Let X be a finite CW complex, and let $p : \tilde{X} \to X$ be a (finite) d-sheeted cover. Then $\chi(\tilde{X}) = d\chi(X)$.

(e) **(Topology Qual, Aug 2020).** Show that a finite group G of order 7 cannot act freely on $\mathbb{C}\mathrm{P}^5$.

(f) **(Topology Qual, Jan 2016).** Let Σ_g be a compact connected surface of genus g. Let $f : \Sigma_g \to \Sigma_3$ be a covering space. Show that g must be odd.

3. (a) Let X be a CW complex constructed by taking a k-sphere S^k and attaching a $(k+1)$-cell e^{k+1} by a degree-n attaching map $S^k \to S^k$. Compute the homology of X.

Your answer gives a solution to:

(Topology Qual, Sep 2016). For given $k, n \geq 1$, construct a topological space M such that $\widetilde{H}_k(M) = \mathbb{Z}/n$ and $\widetilde{H}_k(M) = 0$ for all $i \neq k$.

(b) **Definition (Moore space).** Fix an abelian group G and an integer $p \geq 1$. A *Moore space* $M(G, p)$ is a space such that

$$\widetilde{H}_k(M(G,p)) = \begin{cases} G, & k = p \\ 0, & k \neq p. \end{cases}$$

Let G be a finitely generated abelian group. Explain how to use part (a) and wedge sums to construct a Moore space $M(G, p)$.

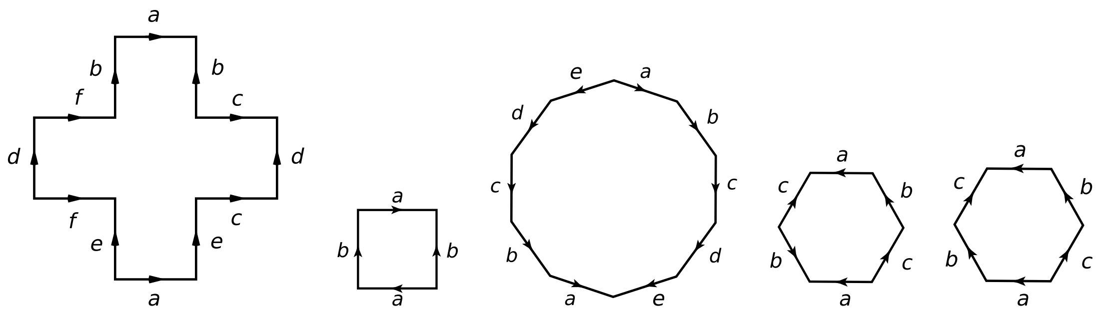
Remark: We can take a wedge sum of Moore spaces to construct a space X with specified homology groups in every homological degree.

4. **(Classification of surfaces)**.

(a) Read Tai-Danae Bradley's post on the classification of closed surfaces.

`https://www.math3ma.com/blog/classifying-surfaces`

Use the instructions to classify the following surfaces.

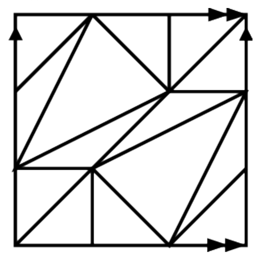


(b) **Definition (Triangulation).** A *triangulation* or *simplicial complex structure* on a space X is a Δ-complex structure that satisfies an additional condition: if two simplices σ_1 and σ_2 intersect, then their intersection must be a single common subsimplex of σ_1 and σ_2.

It is a nontrivial fact (we will not prove) that every compact surface admits a triangulation. Read the following proof of the classification theorem for orientable surfaces.

`https://www3.nd.edu/~andyp/notes/ClassificationSurfaces.pdf`

Summarize the proof, illustrating the steps in the case of the following triangulation of a torus:



5. **(Homology with coefficients).**

Definition (Homology with coefficients in G). Let G be an abelian group, and let X be a space. Then we define the *homology of X with coefficients in G* to be the homology of the chain complex $(C_*(X;G), \partial)$, defined as follows. The nth chain group $C_n(X;G)$ is defined to be the group of formal linear combinations of singular n-simplices in X with coefficients in G,

$$C_*(X;G) = \left\{ \sum_i n_i \sigma_i \;\middle|\; \sigma_i \text{ a singular } n\text{-simplex in } X,\, n_i \in G \right\}.$$

The differential

$$\partial_n : C_n(X;G) \to C_{n-1}(X;G)$$

is defined by the formula

$$\partial_n \left(\sum_i n_i \sigma_i \right) = \sum_{i,j} n_i (-1)^j \sigma_i|_{[v_0, \ldots, \hat{v_j}, \ldots, v_n]}.$$

In the case $G = \mathbb{Z}$, observe that this definition is our usual definition of singular homology.

Just as with singular homology, we can define relative homology by setting

$$C_n(X, A; G) = C_n(X;G)/C_n(A;G)$$

and we can define reduced homology as the homology of the augmented chain complex

$$\cdots \xrightarrow{\partial_2} C_1(X, A; G) \xrightarrow{\partial_1} C_0(X, A; G) \xrightarrow{\epsilon} G \longrightarrow 0$$

All of our major results on homology (long exact sequence of a pair, Mayer–Vietoris, relation to simplicial homology, relation to cellular homology, etc) hold for homology with coefficients, and the proofs carry over with minimal modification.

(a) For a Δ-complex X, formulate a definition of simplicial homology for X with coefficients in $G = \mathbb{Z}/2\mathbb{Z}$. (You do not need to prove this, but it will agree with the singular homology groups with coefficients $H_*(X;\mathbb{Z}/2\mathbb{Z})$.) Working directly from your definition, compute the homology groups of $\mathbb{R}\mathrm{P}^2$ with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

(b) **(The universal coefficient theorem for homology).** Let X be a space, and G an abelian group. For each n there is a short exact sequence of abelian groups

$$0 \longrightarrow H_n(X) \otimes_{\mathbb{Z}} G \longrightarrow H_n(X;G) \longrightarrow \mathrm{Tor}(H_{n-1}(X), G) \longrightarrow 0.$$

The sequence splits (though not canonically).

We will not prove this theorem (or even define the functor Tor), but you can assume the following results.

Proposition (Some properties of Tor). Let A, A_i, and B be abelian groups. The Tor functor satisfies the following.

(i) $\mathrm{Tor}(A,B) = \mathrm{Tor}(B,A)$
(ii) $\mathrm{Tor}\left(\bigoplus_i A_i, B\right) = \bigoplus_i \mathrm{Tor}(A_i, B)$
(iii) $\mathrm{Tor}(A,B) = 0$ if A or B is torsion-free
(iv) $\mathrm{Tor}(A,B) = \mathrm{Tor}(T(A),B)$, where $T(A)$ is the torsion subgroup of A.
(v) $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, B) = \ker(B \xrightarrow{n} B)$

Use these results to compute the $\mathbb{Z}/2\mathbb{Z}$-homology of $\mathbb{R}\mathrm{P}^n$.

6. **Bonus (Orientable manifolds).** You may read Hatcher "Orientations and Homology" (p233) while you complete this question.

Definition (Local orientation). Let M be an n-dimensional manifold. A *local orientation* of M at a point $x \in M$ is a choice of generator μ_x of the infinite cyclic group $H_n(M, M \setminus \{x\})$.

Definition (Orientation). Let M be an n-dimensional manifold. An *orientation* of M is a function $x \mapsto \mu_x$ assigning to each point in M a local orientation $\mu_x \in H_n(M, M\setminus\{x\})$, subject to the following condition: each $x \in M$ has a neighbourhood $\mathbb{R}^n \subseteq M$ containing an open ball B of finite radius about x such that the local orientation μ_y at each $y \in B$ are the images of one generator $\mu_B \in H_n(M, M\setminus B)$ under the natural maps $H_n(M, M\setminus B) \to H_n(M, M\setminus\{y\})$. If an orientation exists, M is *orientable*.

Remark: Let R be a ring. There is a notion of a (local) R-orientation of a manifold M using the analogous definitions with homology with coefficients in R. The theory is particularly important when we let $R = \mathbb{Z}/2\mathbb{Z}$. In fact, every manifold is $\mathbb{Z}/2\mathbb{Z}$-orientable!

(a) **Definition (Orientation double-cover).** Let

$$\widetilde{M} = \{\mu_x \mid x \in M, \mu_x \text{ is a local orientation of } M \text{ at } x\}.$$

We topologize $\widetilde{M}$ as follows: given an open ball $B \subseteq \mathbb{R}^n \subseteq M$ of finite radius and generator $\mu_B \in H_n(M, M \setminus B)$, let $U(\mu_B)$ be the set of all $\mu_x \in \widetilde{M}$ such that $x \in B$ and μ_x is the image of μ_B under the natural map $H_n(M, M\setminus B) \to H_n(M, M\setminus\{x\})$. Then $U(\mu_B)$ is a basis for a topology on M.

Verify that $U(\mu_B)$ is a basis for a topology on $\widetilde{M}$.

(b) Prove that the map

$$\begin{aligned} \widetilde{M} &\longrightarrow M \\ \mu_x &\longmapsto x \end{aligned}$$

is a 2-sheeted cover of M. It is the *orientation double-cover* of M .

(c) Assume M is connected. Prove that $\widetilde{M}$ has two components if and only if M is orientable, and one component if and only if M is nonorientable.

(d) Deduce that any simply connected manifold is orientable, and more generally a connected manifold M is orientable if $\pi_1(M)$ has no subgroup of index 2.

(e) Prove that $\widetilde{M}$ is orientable.

Remark: The following is a very important result about manifolds.

> **Definition / Theorem (Fundamental class).** Let M be a connected, oriented, compact manifold without boundary of dimension n. Then its top homology group is infinite cyclic:
>
> $$H_n(M) \cong \mathbb{Z}.$$
>
> A choice of generator for $H_n(M)$ is called a *fundamental class* of M.
>
> If M is a connected compact manifold without boundary that is not orientable, then $H_n(M) = 0$.

Let M be a connected, oriented, compact manifold without boundary of dimension n. If M has a finite triangulation, a fundamental class μ will be the sum of its (appropriately oriented) top-dimensional simplices.

The natural map $H_n(M) \to H_n(M, M \setminus \{x\})$ is an isomorphism for each $x \in M$, so a choice of fundamental class defines an orientation of M.

Remark: Let M be a compact orientable manifold. Mathematicians once wondered whether every homology class of M was the image of a fundamental class of some oriented submanifold of M. Unfortunately this is not true, but Thom proved it is close to true. This gives us a very concrete way to understand the homology classes of M!

Wellbeing

(This section is completely optional. This is a nudge to prioritize your wellbeing during the pandemic.)

1. **(Health comes first).** This week, make your health and wellness goals a priority.
2. **(Looking forward).**
   (a) This week, brainstorm some goals for the summer. These can be academic, professional, personal, social, spiritual, heath/wellness, . . . Can you make goals SMART?
   (b) Think of some activities you are looking forward to doing over the summer break. The summer will be here soon!