Terms and concepts covered: Categories; objects; morphisms; examples. Monic and epic morphisms. Covariant / contravariant functors. Universal property. Free groups: construction and universal property. Path, homotopy of paths, composition of paths. Reparameterization. Loops, basepoint, fundamental group $\pi_1(X, x_0)$ of X based at x_0.

Corresponding reading: Any reference on basic category theory (like Wikipedia or Tai-Danae Bradley's blog). Hatcher Ch 1.1, "Paths and Homotopy","Fundamental group of the circle".)

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. **(Monic and epic morphisms).**
   (a) Consider the category of sets, the category of abelian groups, and the category of topological spaces. Prove that in these categories, a morphism is monic if and only if it is a injective map.
   (b) Consider the category of sets, the category of abelian groups, and the category of topological spaces. Prove that in these categories, a morphism is epic if and only if it is a surjective map.
   (c) Prove that in the category of rings, the map $\mathbb{Z} \to \mathbb{Q}$ is an epic morphism that is not surjective.
2. **Definition (Isomorphism).** Let $\mathscr{C}$ be a category. A morphism $f : X \to Y$ in $\mathscr{C}$ is an *isomorphism* if there exists a morphism $g : Y \to X$ in $\mathscr{C}$ such that $f \circ g = Id_Y$ and $g \circ f = Id_X$. Then we write $g = f^{-1}$, and we say that the objects X and Y are *isomorphic*.
   (a) Verify that this definition is agrees with your notion of "isomorphism" in every context you have encountered it.
   (b) Recall that the *homotopy category* $\underline{\text{hTop}}$ is the category of topological spaces and homotopy classes of continuous maps. Verify that an isomorphism in this category is precisely a homotopy equivalence.
   (c) Verify that "isomorphism" is an equivalence relation on objects in $\mathscr{C}$.
   (d) Let $\mathscr{C}$ be a category containing objects A and B, and let F be a functor $F : \mathscr{C} \to \mathscr{D}$. Show that if A and B are isomorphic objects of $\mathscr{C}$, then $F(A)$ and $F(B)$ will be isomorphic objects of $\mathscr{D}$.
3. **(Groups as categories).** Given a group G, define a category $\mathscr{G}$ with a single object $\bigstar$ and morphisms $\text{Hom}_{\mathscr{G}}(\bigstar, \bigstar) = \{g \mid g \in G\}$. The composition law is given by the group operation.
   (a) Show that a function between groups $G \to H$ is a group homomorphism if and only if the corresponding map between categories $\mathscr{G} \to \mathscr{H}$ is a functor.
   (b) (For those who have studied group representations). For a field k, let k–$\underline{\text{vect}}$ be the category of k-vector spaces and k-linear maps. Show that the definition of a functor from $\mathscr{G}$ to k–$\underline{\text{vect}}$ is equivalent to the definition of a linear representation of G over k.
4. **(Power set functors).** Let $\underline{\text{f Set}}$ denote the category of finite sets and all functions between sets. Let $\mathscr{P} : \underline{\text{f Set}} \to \underline{\text{f Set}}$ be the function that takes a finite set A to its *power set* $\mathscr{P}(A)$, the set of all subsets of A. If $f : A \to B$ is a function of finite sets, let $\mathscr{P}(f) : \mathscr{P}(A) \to \mathscr{P}(B)$ be the function that takes a subset $U \subseteq A$ to the subset $f(U) \subseteq B$.
   (a) Show that $\mathscr{P}$ is a covariant functor.
   (b) What if we had instead defined $\mathscr{P}(f) : \mathscr{P}(B) \to \mathscr{P}(A)$ to take a subset $U \subseteq B$ to its preimage $f^{-1}(U) \subseteq A$ under f?
5. **(Open subsets functor)** Let $\underline{\text{Top}}$ be the category of topological spaces and continuous maps. Let $\underline{\text{Set}}$ be the category of sets and all functions of sets. Define a *contravariant* functor $\mathcal{O} : \underline{\text{Top}} \to \underline{\text{Set}}$ that takes a topological space X to its collection $\mathcal{O}(X)$ of open subsets. How should we define $\mathcal{O}$ on morphisms to make it well-defined and functorial?

6. **(More adjoints).** Let $\underline{\text{Top}}$ be the category of topological spaces and continuous maps. Let $\underline{\text{Set}}$ be the category of sets and all functions of sets. Let $\mathcal{F}$ be the "forgetful map"

$$\mathcal{F} : \underline{\text{Top}} \longrightarrow \underline{\text{Set}}$$

that takes a space X to its underlying set. Define maps

$$I, D : \underline{\text{Set}} \longrightarrow \underline{\text{Top}}$$

so that for a set S, $D(S)$ is the set S with the discrete topology, and $I(S)$ is the set S with the indiscrete topology. Prove that there are bijections

$$\text{Hom}_{\underline{\text{Set}}}(A, \mathcal{F}(X)) \cong \text{Hom}_{\underline{\text{Top}}}(D(A), X)$$

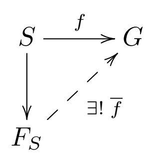
and

$$\text{Hom}_{\underline{\text{Set}}}(\mathcal{F}(X), A) \cong \text{Hom}_{\underline{\text{Top}}}(X, I(A)).$$

It turns out that these bijections are "natural", so this result shows that D is a *left adjoint* to $\mathcal{F}$, and I is the *right adjoint* to $\mathcal{F}$.

7. **(Constructing the free group).** In class, we constructed the free group F_S on a set S. Verify that our construction does indeed satisfy the universal property of the free group.

8. **(Free abelian groups).** Recall the universal property of the free group F_S on a set S: given any group G and any map of sets $f : S \to G$, the map f extends uniquely to a group homomorphism $\overline{f} : F(S) \to G$. In other words, there is a unique homomorphism $\overline{f}$ making the following diagram commute.



(a) Consider the same universal property in the category of abelian groups (so now G must be abelian). Show that the universal property defines the free abelian group on S, that is, $F_S \cong \bigoplus_S \mathbb{Z}$.

(b) Why doesn't the free abelian group on S satisfy the universal property in the category of groups?

9. **(Homotopies of paths define an equivalence relation).** Let X be a space, and $x_0, x_1 \in X$. Consider all paths $\gamma : I \to X$ satisfying $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Show that the relation of being path homotopic (ie, homotopic rel $\{0,1\}$) is an equivalence relation on these paths.

10. **(Homotopy of paths respects composition of paths).**

(a) Show that homotopy of paths is compatible with composition of paths. In other words, suppose we have points x_0, x_1, x_2 in a space X. Suppose that paths α and α' from x_0 to x_1 are homotopic rel $\{0,1\}$, and suppose that paths β and β' from x_1 to x_2 are homotopic rel $\{0,1\}$. Verify that the paths $\alpha \cdot \beta$ and $\alpha' \cdot \beta'$ from x_0 to x_2 are homotopic rel $\{0,1\}$.

(b) What would happen if we just considered the paths α and β up to homotopy (instead of homotopy rel $\{0,1\}$)? Would homotopy still respect composition of paths?

11. **(Loop spaces).** For a topological space X with basepoint x_0, let ΩX denote the set of loops in X based at x_0. The loop space ΩX has a binary operation given by composition of loops. Explain why (in general) ΩX fails to be a group with this operation, by considering whether each of the associativity, identity, and inverse axioms will hold on the level of loops (in contrast to "loops up to path homotopy").

12. **(Paths in $\mathbb{R}^n$).**

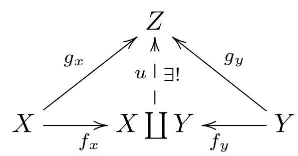
(a) Let $\gamma : I \to \mathbb{R}^n$ be a path from x_0 to x_1. Use the straight-line homotopy to show that γ is homotopic rel $\{0,1\}$ to any other path in $\mathbb{R}^n$ from x_0 to x_1.

(b) Deduce that $\pi_1(\mathbb{R}^n, 0)$ is the trivial group.

Assignment questions

(Hand these questions in!)

1. **(Coproducts).** Let $\mathcal{C}$ be a category with objects X and Y. The *coproduct* of X and Y (if it exists) is an object $X \coprod Y$ in $\mathcal{C}$ with maps $f_x : X \to X \coprod Y$ and $f_y : Y \to X \coprod Y$ satisfying the following universal property: whenever there is an object Z with maps $g_x : X \to Z$ and $g_y : Y \to Z$, there exists a unique map $u : X \coprod Y \to Z$ that makes the following diagram commute:



   (a) Let X and Y be objects in $\mathcal{C}$. Show that, if the coproduct $(X \coprod Y, f_x, f_y)$ exists in $\mathcal{C}$, then the universal property determines it uniquely up to unique isomorphism.

   (b) Explain how to reinterpret this universal property as a bijection of sets

$$\mathrm{Hom}_{\mathcal{C}}(X \coprod Y, Z) \cong \mathrm{Hom}_{\mathcal{C}}(X, Z) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z)$$

   for objects X, Y, Z.

   (c) Prove that in the category of sets, the coproduct $X \coprod Y$ of sets X and Y is their disjoint union.

   (d) Let Top be the category of topological spaces and continuous maps. The coproduct of $X \coprod Y$ of spaces X and Y is called the *(topological) disjoint union*. The underlying set is the disjoint union. Describe the topology on the disjoint union that satisfies the universal property.

   (e) Prove that in the category of abelian groups, the coproduct of groups $X \coprod Y$ is $X \oplus Y$ with the canonical inclusions of X and Y. In other words, this universal property defines the direct sum operation on abelian groups.

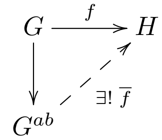
   (f) In the category Grp of groups, the univeral property for the coproduct does *not* define the direct product operation. The coproduct $G \coprod H$ of groups G and H is a construction called the *free product* of G and H, and denoted $G * H$. Determine how to construct the group $G * H$ along with maps $G \to G * H$ and $H \to G * H$ that satisfy the universal property.
   Hint: The coproduct $\mathbb{Z} * \mathbb{Z}$ is the free group on two generators.

2. **(Abelianization).** Let Grp denote the category of groups and group homomorphisms, and let Ab denote the category of abelian groups and group homomorphisms. Define the *abelianization* G^{ab} of a group G to be the quotient of G by its *commutator subgroup* $[G, G]$, the subgroup normally generated by *commutators*, elements of the form $ghg^{-1}h^{-1}$ for all $g, h \in G$.

   (a) Define a map of categories $[-,-] : \underline{\mathrm{Grp}} \to \underline{\mathrm{Grp}}$ that takes a group G to its commutator subgroup $[G, G]$, and a group morphism $f : G \to H$ to its restriction to $[G, G]$. Check that this map is well defined (ie, check that $f([G, G]) \subseteq [H, H]$) and verify that $[-,-]$ is a functor.

   (b) Show that G^{ab} is an abelian group. Show moreover that if G is abelian, then $G = G^{ab}$.

   (c) Show that the quotient map $G \to G^{ab}$ satisfies the following universal property: Given any **abelian** group H and group homomorphism $f : G \to H$, there is a unique group homomorphism $\overline{f} : G^{ab} \to H$ that makes the following diagram commute:



   This universal property shows that G^{ab} is in a sense the "largest" abelian quotient of G.

(d) Show that the map ab that takes a group G to its abelianization G^{ab} can be made into a functor $ab : \underline{\text{Grp}} \to \underline{\text{Ab}}$ by explaining where it maps morphisms of groups $f : G \to H$, and verifying that it is functorial.

(e) The category $\underline{\text{Ab}}$ is a subcategory of $\underline{\text{Grp}}$. Define the functor $\mathcal{A} : \underline{\text{Ab}} \to \underline{\text{Grp}}$ to be the inclusion of this subcategory; $\mathcal{A}$ takes abelian groups and group homomorphisms in $\underline{\text{Ab}}$ to the same abelian groups and the same group homomorphisms in $\underline{\text{Grp}}$. Briefly explain why the universal property in Part (c) can be rephrased as follows: Given groups $G \in \underline{\text{Grp}}$ and $H \in \underline{\text{Ab}}$, there is a natural bijection between the sets of morphisms:

$$\text{Hom}_{\underline{\text{Grp}}}(G, \mathcal{A}(H)) \cong \text{Hom}_{\underline{\text{Ab}}}(G^{ab}, H)$$

Remark: Since this bijection is "natural" (a condition we won't formally define or check) it means that $\mathcal{A} : \underline{\text{Ab}} \to \underline{\text{Grp}}$ and $ab : \underline{\text{Grp}} \to \underline{\text{Ab}}$ are what we call a pair of *adjoint functors*.

3. *Hint:* These results are proved in Hatcher Ch 1.1. You may read their proofs there, but then put the book away and write your solutions independently!

(a) **(Reparameterization preserves homotopy class).**

Definition (Reparameterization). Let $\gamma : I \to X$ be a path in a space X. A *reparameterization* of γ is a path $\gamma \circ \phi$ obtained by precomposing γ by a map $\phi : I \to I$ such that $\phi(0) = 0$ and $\phi(1) = 1$.

Show that γ and any reparameterization $\gamma \circ \phi$ are homotopic ref $\{0, 1\}$.

(b) **(The fundamental group is a group)**. For a space X with basepoint x_0, we defined the fundamental group $\pi_1(X, x_0)$ to be the group of loops in X based on x_0 up to path homotopy, under composition of paths. Complete our proof that this is a group, by verifying the following. Let c be the constant loop at x_0, and let $\gamma, \gamma_1, \gamma_2, \gamma_3$ be any loops based at x_0.

- Associativity: $\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)$ is a reparameterization of $(\gamma_1 \cdot \gamma_2) \cdot \gamma_3$.
- Identity: $\gamma \cdot c$ is a reparameterization of γ. (A similar argument shows $c \cdot \gamma \simeq \gamma$).
- Inverses: $\gamma \cdot \overline{\gamma} \simeq c$, where $\overline{\gamma}(t) = \gamma(1-t)$. (A similar argument shows $\overline{\gamma} \cdot \gamma \simeq c$).

(c) **($\pi_1(X)$ is well-defined for path-connected X)**. Prove the following.

Theorem (Change of basepoint). Let X be a space, and let x_0 and x_1 be two points in X connected by a path h. Then the *change-of-basepoint map*

$$\begin{aligned} \pi_1(X, x_1) &\longrightarrow \pi_1(X, x_0) \\ [\gamma] &\longmapsto [h \cdot \gamma \cdot \overline{h}] \end{aligned}$$

is an isomorphism. Here, $\overline{h}$ is defined as the path $\overline{h}(s) = h(1-s)$.

Conclude that (up to isomorphism) the fundamental group of X does not depend on the choice of basepoint, only on the choice of path component of the basepoint. If X is path-connected, it now makes sense to refer to "the" fundamental group of X and write $\pi_1(X)$ for the abstract group.

4. The goal of this question is to prove this theorem.

Theorem (The fundamental group of S^1). Let S^1 denote the unit circle in $\mathbb{R}^2$. There is an isomorphism

$$\begin{aligned} \Phi : \mathbb{Z} &\longrightarrow \pi_1\Big(S^1, (1,0)\Big) \\ n &\longmapsto [\omega_n : t \mapsto (\cos(2\pi n t), \sin(2\pi n t))]. \end{aligned}$$

Hint: Hatcher proves this result in Theorem 1.7, using an approach that is closely related but not identical to the one below. If you read Hatcher's proof, please put the book away as you write your own solutions.

(a) Verify that $\Phi(m+n)$ and $\Phi(m)\cdot\Phi(n)$ are homotopic, so Φ is a group homomorphism.

(b) **Definition (Covering map).** Let $p : E \to B$ be a continuous map of topological spaces. The map p is called a *covering map* if every point $b \in B$ has some neighbourhood U_b with the following property. The preimage $p^{-1}(U_b) \subseteq E$ is the union of disjoint open sets $\{V_{b,\alpha}\}$ in E such that for each α the restriction $p|_{V_{b,\alpha}}$ is a homeomorphism from $V_{b,\alpha}$ to U_b. In this case, E is called a *covering space* of B.

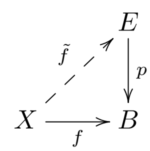
Prove that the map

$$\begin{aligned} p : \mathbb{R} &\longrightarrow S^1 \\ x &\longmapsto (\cos(2\pi x), \sin(2\pi x)) \end{aligned}$$

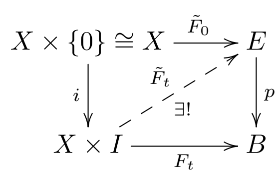
is a covering map.

(c) The following *homotopy lifting property* is a crucial feature of covering maps. We will prove it later in the course.

Definition (Lift). Let $p : E \to B$ be a covering map, and let $f : X \to B$ be a continuous map. A *lift* of f is a map $\tilde{f} : X \to E$ such that $p \circ \tilde{f} = f$.



Theorem (Covering maps have the homotopy lifting property). Let $p : E \to B$ be a covering map, and let $F_t : X \times I \to B$ be a homotopy of maps $X \to B$. Then given any lift $\tilde{F}_0 : X \to E$ of F_0, there exists a unique lift $\tilde{F}_t : X \times I \to E$ of F_t whose restriction to $t = 0$ is the lift $\tilde{F}_0$.



Note that this theorem gives both *existence* and *uniqueness* of $\tilde{F}_t$. Briefly explain why the theorem implies the following two results.

(i) For each path $\gamma : I \to S^1$ starting at $(1,0)$ and each $x \in p^{-1}(1,0)$ there is a unique lift $\tilde{\gamma} : I \to \mathbb{R}$ starting at x .

(ii) Let $F_t : I \times I \to S^1$ be a homotopy rel $\{0,1\}$ starting at $(1,0) \in S^1$. For each $x \in p^{-1}(1,0)$, there is a unique homotopy $\tilde{F}_t : I \times I \to \mathbb{R}$ with $\tilde{F}_0$ a path starting at $x \in \mathbb{R}$.

(d) Explain why the homotopy lifting property implies that the lifted homotopy $\tilde{F}_t$ in (ii) must be a homotopy rel $\{0,1\}$. *Hint:* Consider the paths $t \mapsto F_t(0)$ and $t \mapsto F_t(1)$.

(e) Describe the path $\tilde{\omega}_n : I \to \mathbb{R}$ starting at $0 \in \mathbb{R}$ that lifts the loop

$$\begin{aligned} \omega_n : I &\longrightarrow S^1 \\ t &\longmapsto (\cos(2\pi n t), \sin(2\pi n t)), \end{aligned}$$

and describe the class of paths in $\mathbb{R}$ that are homotopic rel $\{0,1\}$ to $\tilde{\omega}_n$.

(f) Prove that Φ is surjective and injective, hence an isomorphism.

Wellbeing

(This section is completely optional. This is a nudge to prioritize your wellbeing during the pandemic.)

1. **(Health comes first).** Prioritize your sleep, fitness, nutrition, and other health goals.

2. **(Shared experiences).** Sometime this week, arrange to share an activity with a friend or family member, even if it is over the phone or Zoom. Go on walks at the same time, cook together, have a meal together, watch the same TV show together. Or, if you cannot find time, just turn on Zoom as you quietly work together.