Terms and concepts covered: $\pi_1(S^n)$, π_1 of a product, π_1 as a functor.

Corresponding reading: Hatcher Ch 0, "Homotopy extension property". Hatcher Ch 1.1, "Induced homomorphisms".

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let Grp be the category of groups. Consider the map $Z : \underline{\text{Grp}} \to \underline{\text{Grp}}$ that takes every group G to itself and every morphism f to the zero map. Is Z a functor?

2. **Definition (Opposite category).** Let $\mathscr{C}$ be a category. The *opposite category* $\mathscr{C}^{op}$ is a category defined as follows: The objects of $\mathscr{C}^{op}$ are the same as the objects of $\mathscr{C}$. For objects $X, Y \in \mathscr{C}^{op}$, the morphisms are
$$\text{Hom}_{\mathscr{C}^{op}}(X,Y) = \text{Hom}_{\mathscr{C}}(Y,X).$$
The composite $f \circ g$ of morphisms f, g in $\mathscr{C}^{op}$ is defined to be the morphism $g \circ f$ in $\mathscr{C}$.

   Informally, $\mathscr{C}^{op}$ is the category $\mathscr{C}$ after "reversing all the arrows". Show that the definition of a contravariant functor $\mathscr{C} \to \mathscr{D}$ is equivalent to the definition of a covariant functor $\mathscr{C}^{op} \to \mathscr{D}$.

3. Let $\mathscr{C}$ be a *locally small* category. ("Locally small" is a condition to deal with set-theoretic issues. All the categories we encounter will have this property). For each object $A \in \mathscr{C}$, we can define two *hom functors* from $\mathscr{C}$ to Set,
$$\text{Hom}_{\mathscr{C}}(A,-) \qquad \text{and} \qquad \text{Hom}_{\mathscr{C}}(-,A).$$
The first is covariant and the second is contravariant. They are defined as follows.
$$\begin{aligned} \text{Hom}_{\mathscr{C}}(A,-) : \mathscr{C} &\longrightarrow \underline{\text{Set}} \\ B &\longmapsto \text{Hom}_{\mathscr{C}}(A,B) \\ [f : B \to C] &\longmapsto \left[f_* : \begin{array}{ll} \text{Hom}_{\mathscr{C}}(A,B) & \to \text{Hom}_{\mathscr{C}}(A,C)] \\ \phi & \mapsto f \circ \phi \end{array} \right] \end{aligned}$$
$$\begin{aligned} \text{Hom}_{\mathscr{C}}(-,A) : \mathscr{C} &\longrightarrow \underline{\text{Set}} \\ B &\longmapsto \text{Hom}_{\mathscr{C}}(B,A) \\ [f : B \to C] &\longmapsto \left[f^* : \begin{array}{ll} \text{Hom}_{\mathscr{C}}(C,A) & \to \text{Hom}_{\mathscr{C}}(B,A)] \\ \phi & \mapsto \phi \circ f \end{array} \right] \end{aligned}$$
   (a) Verify that, for each object $A \in \mathscr{C}$, the maps $\text{Hom}_{\mathscr{C}}(A,-)$ and $\text{Hom}_{\mathscr{C}}(-,A)$ are functors.

   (b) Explain the sense in that the forgetful functor $\underline{\text{Top}} \to \underline{\text{Set}}$ is "the same"* as the functor $\text{Hom}_{\underline{\text{Top}}}(\{*\},-)$. (*Technically, they are *naturally isomorphic* functors).

   (c) Explain the sense in that the forgetful functor $\underline{\text{Grp}} \to \underline{\text{Set}}$ is "the same" as the functor $\text{Hom}_{\underline{\text{Grp}}}(\mathbb{Z},-)$.

   (d) Fix a field k. Consider the forgetful functor k–$\underline{\text{vect}} \to \underline{\text{Set}}$. Is there is k-vector space V so that this functor is "the same" as $\text{Hom}_{k-\underline{\text{vect}}}(V,-)$?

   If a functor $F : \mathscr{C} \to \underline{\text{Set}}$ is naturally isomorphic to a hom functor, then F is called *representable*.

4. Let H be a group containing a subset S, and let $S \hookrightarrow H$ be the inclusion. In this question we investigate why, in general, H (along with the map $S \hookrightarrow H$) could fail to satisfy the universal property of the free group on S.

   (a) Suppose S generates H. Prove H satisfies the "uniqueness" condition of the universal property.

(b) Show by example that, if S does not generate H, then H could fail to satisfy the "uniqueness" condition of the universal property.

(c) Suppose that the elements of S satisfy some *relations* (a term we will define formally later in course). For example, the elements of S could commute, or might have finite order. Show that H will fail the "existence" condition of the universal property.

5. (a) Let $f : X \to Y$ and $g : Y \to X$ be maps of sets, and suppose that $f \circ g = id_Y$. Show that f is surjective, and g is injective.

   (b) Let $f : X \to Y$ and $g : Y \to X$ be morphisms in a category $\mathcal{C}$ such that $f \circ g = id_Y$. Show that f is an epimorphism, and g is a monomorphism.

   (c) Again let $f : X \to Y$ and $g : Y \to X$ be morphisms in a category $\mathcal{C}$ such that $f \circ g = id_Y$. Show moreover that the images of f and g under any covariant functor must also be an epimorphism and a monomorphism, respectively.

6. Let X be a contractible space.

   (a) Show that X is path-connected.

   (b) Show that $\pi_1(X)$ is the trivial group.

   (c) Conclude that the S^1, the torus, and in general the n-torus are not contractible, nor is any product of the form $S^1 \times Y$.

7. (a) Prove that a map of spaces

$$\begin{aligned} Z &\longrightarrow X \times Y \\ z &\longmapsto (f_X(z), f_Y(z)) \end{aligned}$$

   is continuous if and only if the component maps $f_X : Z \to X$ and $f_Y : Z \to Y$ are continuous.

   (b) Describe the bijection between paths in $X \times Y$ and pairs of paths in X and in Y. Similarly, describe the bijection between homotopies of maps in $X \times Y$ and pairs of homotopies in X and in Y.

   (c) Check the details of our proof that

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

8. (a) Given a pair of continuous maps $f_1 : Z_1 \to W_1$ and $f_2 : Z_2 \to W_2$, show that their product is continuous,

$$\begin{aligned} f_1 \times f_2 : Z_1 \times Z_2 &\longrightarrow W_1 \times W_2 \\ (z_1, z_2) &\longmapsto \big(f_1(z_1), f_2(z_2)\big) \end{aligned}$$

   (b) Given homotopy equivalences of spaces $X_1 \simeq Y_1$ and $X_2 \simeq Y_2$, show that there is a homotopy equivalence $X_1 \times X_2 \simeq Y_1 \times Y_2$.

9. (a) Suppose that $f : X \to Y$ is a homeomorphism. Show that, for any subset $A \subseteq X$, f induces a homeomorphism $f|_{X-A} : (X - A) \to (Y - f(A))$.

   (b) Show that $\mathbb{R}^1$ is not homeomorphic to $\mathbb{R}^n$ for any $n > 1$.
   Hint: Consider the path components of $\mathbb{R}^1 - \{0\}$.

   (c) Show that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for any $n > 2$.
   Hint: Consider the fundamental group of $(\mathbb{R}^2 - \{0\}) \cong \mathbb{R} \times S^1$.

10. For a continuous map $f : X \to Y$, we defined the induced map on fundamental groups

$$\begin{aligned} f_* : \pi_1(X, x_0) &\longrightarrow \pi_1(Y, f(x_0)) \\ [\gamma] &\longmapsto [f \circ \gamma]. \end{aligned}$$

   Complete our proof that π_1 is a functor by checking

- f_* is well-defined on homotopy classes
- f_* is a group homomorphism
- $(id_X)_* = id_{\pi_1(X,x_0)}$
- $(f \circ g)_* = (f_*) \circ (g_*)$

11. Draw a collection of finite graphs (in the sense of graph theory). In each graph G, identify a maximal tree T. Verify that the quotient G/T is a wedge of 1-spheres. Use Assignment Problem 4 to explain why the quotient map $G \to G/T$ is a homotopy equivalence.

12. Let X be a CW complex. Consider the interval I as a CW complex with two vertices and one edge. Describe the natural CW complex structure on $X \times I$. What is its n-skeleton, in terms of the skeleta of X?

Assignment questions

(Hand these questions in!)

1. In this question, we will develop some applications of our calculation $\pi_1(S^1) \cong \mathbb{Z}$.

   (a) **Definition (Retraction).** Let X be a topological space, and $A \subseteq X$ a subspace. A *retraction* $r : X \to A$ is a continuous map such that $r(a) = a$ for all $a \in A$. The subspace A is called a *retract* of X.

   (Note: a *deformation retraction* from X to A is a homotopy rel A from id_X to a retraction $r : X \to A$.) Suppose that $r : X \to A$ is a retraction. Let $\iota : A \to X$ denote the inclusion of A. Fix $a \in A$. Show that $\iota_* : \pi_1(A, a) \to \pi_1(X, a)$ is injective, and $r_* : \pi_1(X, a) \to \pi_1(A, a)$ is surjective.
   Hint: Warm-up Questions 5 (a) and 10.

   (b) Explain why no retraction from D^2 to $\partial D^2 = S^1$ can exist.

   (c) Prove the following theorem.

   Theorem (Brouwer fixed-point theorem for D^2). Let $f : D^2 \to D^2$ be a continuous map. Then D^2 has a *fixed point*, that is, there is some $x \in D^2$ such that $f(x) = x$.

   Hint: Suppose $f : D^2 \to D^2$ has no fixed point. Use f to build a retraction from $r : D^2 \to S^1$. (Your map r should be constructed in a way that $r(x)$ depends continuously on the data of x and $f(x)$, but you do not need to prove that r is continuous).

   (d) **(Perron–Frobenius eigenvalues).** Consider a linear map $\mathbb{R}^3 \to \mathbb{R}^3$ for which the corresponding matrix M (with respect to the standard basis) has positive entries. Use the Brouwer fixed point theorem to prove that M has a positive eigenvalue.
   Hint: You may use without proof the fact that the subspace

   $$\{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \geq 0\} \cap S^2 \quad \subseteq \quad \mathbb{R}^3$$

   is homeomorphic to a closed 2-ball.

   (e) Recall that a *vector field* on D^2 is an ordered pair $(x, v(x))$ where $x \in D^2$ and $v(x)$ is a continuous map $v : D^2 \to \mathbb{R}^2$. We view $v(x)$ as a vector based at x. Prove the following theorem.

   Theorem. Given a nonvanishing vector field on D^2, there exists a point $x \in S^1$ where the vector $v(x)$ points radially outward, and a point $y \in S^1$ where the vector $v(y)$ points radially inward.

   Hint: Consider $F_t(x) = tx + (1-t)v|_{S^1}(x)$.

2. (a) Suppose $f_0, f_1 : X \to Y$ are homotopic maps via a homotopy f_t. Let $x_0 \in X$ be a basepoint, and let h be the path $h(t) = f_t(x_0)$. Prove that $\beta_h \circ (f_1)_* = (f_0)_*$, where β_h is the change-of-basepoint map,

   $$\begin{aligned} \beta_h : \pi_1(Y, f_1(x_0)) &\longrightarrow \pi_1(Y, f_0(x_0)) \\ [\gamma] &\longmapsto [h \cdot \gamma \cdot \overline{h}] \end{aligned}$$

(b) Use (a) to deduce that if $f : X \to Y$ is nullhomotopic, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is the trivial map.

(c) Let $f, g : X \to Y$ be homotopic maps, and let

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0)) \quad \text{and} \quad g_* : \pi_1(X, x_0) \to \pi_1(Y, g(x_0))$$

be their induced maps. Use part (a) to show that if f_* is injective, surjective, or trivial, then so is g_*.

(d) Prove the following.

Theorem (π_1 is a homotopy invariant). If $f : X \to Y$ is a homotopy equivalence, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism.

3. Let X be a space, and let $f : S^1 \to X$ be a continuous map. Prove that the following are equivalent.

(i) f is nullhomotopic [the homotopy has no conditions on basepoints].

(ii) View the domain of f as the boundary of D^2. The map f extends to a continuous map $D^2 \to X$.

(iii) The induced map $f_* : \pi_1(S^1) \to \pi_1(X)$ is the zero map.

4. The goal of this question is to prove the following theorem.

Theorem (Collapsing contractible subcomplexes). Let X be a CW complex and $A \subseteq X$ a subcomplex. If A is contractible, then the quotient $X \to X/A$ is a homotopy equivalence.

For this question, you may read Hatcher Chapter 0 "The Homotopy Extension Property" as you write up your solutions.

(a) Define (qualitatively) a deformation retraction from $D^n \times I$ to $(D^n \times \{0\}) \cup (\partial D^n \times I)$.

(b) Let X be a CW complex, and A a subcomplex. Hatcher writes,

"This deformation retraction [from part (a)] gives rise to a deformation retraction of $X^n \times I$ onto $(X^n \times \{0\}) \cup ((X^{n-1} \cup A^n) \times I)$, since $X^n \times I$ is obtained from $(X^n \times \{0\}) \cup ((X^{n-1} \cup A^n) \times I)$ by attaching copies of $D^n \times I$ along $(D^n \times \{0\}) \cup (\partial D^n \times I)$."

Explain this construction in the case that X is the CW complex structure on the 2-disk shown in Figure 1, and A is the left edge. Illustrate (with pictures) the deformation retraction from $X^0 \times I$, $X^1 \times I$, and $X^2 \times I$.

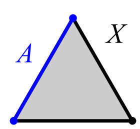
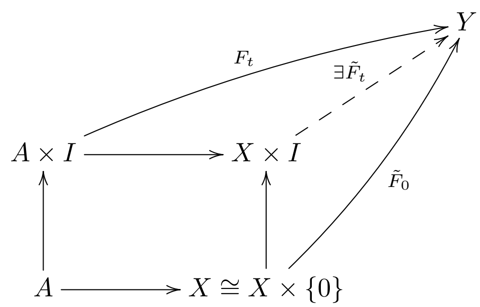


Figure 1: A CW complex structure on D^2

(c) Let A be a subcomplex of a CW complex X. Show that $(X \times \{0\}) \cup (A \times I)$ is a deformation retraction of $X \times I$. *Hint:* Perform the deformation retraction on $X^n \times I$ for the time interval $[\frac{1}{2^{n+1}}, \frac{1}{2^n}]$. See Hatcher Proposition 0.16. You do not need to provide point-set details.

(d) **Definition (Homotopy extension property).** Let X be a topological space and $A \subseteq X$ a subspace. We say that the pair (X, A) has the *homotopy extension property* if, given a homotopy $F_t(a)$ from $A \times I \to Y$ and a map $\tilde{F}_0 : X \to Y$ such that $\tilde{F}_0|_A = F_0$, then there is a homotopy $\tilde{F}_t(x)$ from $X \to Y$ such that $\tilde{F}_t|_A = F_t$. The homotopy $\tilde{F}_t(x)$ is called an *extension* of $F_t(a)$.



The lift $\tilde{F}_t(x)$ need not be unique.

Let A be a subcomplex of a CW complex X. Show that (X, A) has the homotopy extension property. *Hint*: Use the pasting lemma, and the retraction defined by the deformation retraction from part (c) at time $t = 1$.

(e) Read Hatcher Proposition 0.17, which proves our theorem. Explain the steps in the proof and give explicit (if qualitative) descriptions of possibilities for the maps in the case that X is the graph in Figure 2 and A is its central edge.

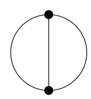


Figure 2: The theta graph

(f) Let X be a space and $A \subseteq X$ a subspace. The *cone* CA on A is the quotient space of $A \times I$ where $A \times \{1\}$ is collapsed to a point. Let Y be the space obtained by gluing CA to X by the identification $(a, 0) \sim a$ for all $a \in A$. Assuming that Y has a CW complex structure for which CA is a subcomplex, briefly explain why $Y \simeq X/A$.

(g) Our theorem does not hold for arbitrary contractible subspaces. Let $X = S^1$ and let A be the complement of a point in S^1, so A is homeomorphic to an open interval. Prove that S^1/A and S^1 are not homotopy equivalent.

Wellbeing

(This section is completely optional. This is a nudge to prioritize your wellbeing during the pandemic.)

1. **(Health comes first).** Make your sleep, fitness, nutrition, and other health goals a priority this week.
2. **(Fresh air and sunlight).** If you're able, get outside every day this week. Dress for the weather! Aim to spend time in the daylight – either outdoors or by a window – at the beginning of your day and before sundown every day.