Terms and concepts covered: Seifert–Van Kampen Theorem, group presentations, π_1 of a graph

Corresponding reading: Hatcher Ch 1.2 and 1.A (up to Proposition 1A.2).

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Show that the free product $*_\alpha G_\alpha$ of trivial groups G_α is trivial.
2. Consider the decomposition of S^1 into two open intervals $S^1 = A \cup B$ as shown in Figure 1. Use this example to show the Van Kampen theorem requires the hypothesis that $A \cap B$ is path-connected.

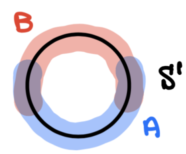

Figure 1: $S^1 = A \cup B$

3. (a) Describe how to construct the n-sphere S^n by gluing two n-disks D^n_A and D^n_B by their boundary along an $(n-1)$-sphere.
   (b) Assume $n \geq 2$. Decompose S^n into a union of open subsets $S^n = A \cup B$, where A is a neighbourhood of the image of D^n_A, and B is a neighbourhood of the image of D^n_B. See Figure 2. Use the Van Kampen theorem to show $\pi_1(S^n) = 0$.

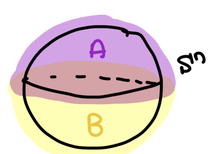

Figure 2: $S^n = A \cup B$

   (c) Where does the proof go wrong when $n = 1$?
4. Find the error in the following flawed "proof" that the circle has trivial fundamental group.

   False proof. Decompose S^1 as a union of open intervals $S^1 = A \cup B \cup C$ as shown in Figure 3. Since A, B, C are open and path-connected, and their pairwise intersections $A \cap B$, $B \cap C$, $A \cap C$

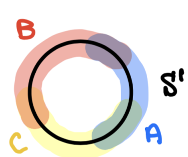

Figure 3: $S^1 = A \cup B$

   are path-connected, it follows that $\pi_1(S^1)$ is a quotient of the free product $\pi_1(A) * \pi_1(B) * \pi_1(C)$. Since A, B, C are contractible, $\pi_1(A) * \pi_1(B) * \pi_1(C)$ is trivial, so $\pi_1(S^1)$ is trivial.

5. (a) Let X be the complement of an open disk in the torus, as in Figure 4. Show that X deformation retracts onto the wedge $S^1 \bigvee S^1$. *Hint:* Consider the flat torus.

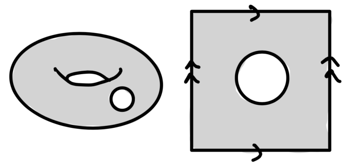

Figure 4: The torus with an open disk deleted

(b) Deduce that $\pi_1(X)$ is the free group on 2 generators.

(c) Apply the Van Kampen theorem to the decomposition of the torus T into open sets shown in Figure 5 to give an alternate computation of $\pi_1(T)$.

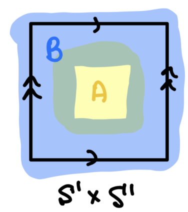

Figure 5: $T = A \cup B$

(d) Apply the Van Kampen theorem to the decomposition of the closed genus-2 surface Σ_2 into open sets shown in Figure 6 to give an alternate computation of $\pi_1(\Sigma_2)$. Compare your solution to the result of Assignment Problem 2.

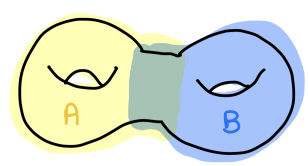

Figure 6: $\Sigma_2 = A \cup B$

6. Why, in the Van Kampen theorem, do we assume that each open set is path-connected?

7. Why, in the Van Kampen theorem, must we assume that each subset in our decomposition is open? *Hint:* Decompose X as a union of points.

8. (a) Let G be a group, and suppose that G is generated by S. Show that the commutator subgroup of G is normally generated by the set

$$\{aba^{-1}b^{-1} \mid a, b \in S\}.$$

(b) Let G_1, G_2, H be groups, and let $f_1 : H \to G_1$ and $f_2 : H \to G_2$ be group homomorphisms. We defined the *free product with amalgamation* $G_1 *_H G_2$ to be the quotient of the free product $G_1 * G_2$ by the normal subgroup generated by the elements

$$\{f_1(h)f_2(h)^{-1} \mid h \in H\}.$$

Show that, if S is a generating set for H, then $G_1 *_H G_2$ is in fact the quotient by the normal subgroup generated by

$$\{f_1(s)f_2(s)^{-1} \mid s \in S\}.$$

9. Use the results of Assignment Question 1 to give a new proof that $\pi_1(S^n) = 0$ for all $n \geq 2$.

Assignment questions

(Hand these questions in!)

1. **(The fundamental group of a CW complex).** In this question, we will continuing developing our program from class on computing the fundamental group of a CW complex.

   (a) **Proposition (The effect of gluing in disks on π_1).** Let X be a path-connected space with basepoint x_0. Let Y be the space obtained from X by gluing in a number of disks D^2_α along their boundary by attaching maps $\varphi_\alpha : \partial D^2_\alpha \to X$. (By abuse of notation, we will also use φ_α to denote the loop $I \to X$ canonically determined by the map $\varphi_\alpha : S^1 \to X$.) For each α, let γ_α be a choice of path from x_0 to $\varphi_\alpha(1,0)$. Then $\pi_1(Y, x_0)$ is the quotient of $\pi_1(X, x_0)$ by the subgroup normally generated by the loops $\{\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma_\alpha}\}_\alpha$.

   This result is proven in Hatcher Proposition 1.26. Explain this proof (with pictures!) in the case that X is a 2-disk with 3 punctures, and Y is constructed by gluing two disks over two punctures via embedddings φ_1, φ_2, as shown in Figure 7. You may read Hatcher while you write your solution.

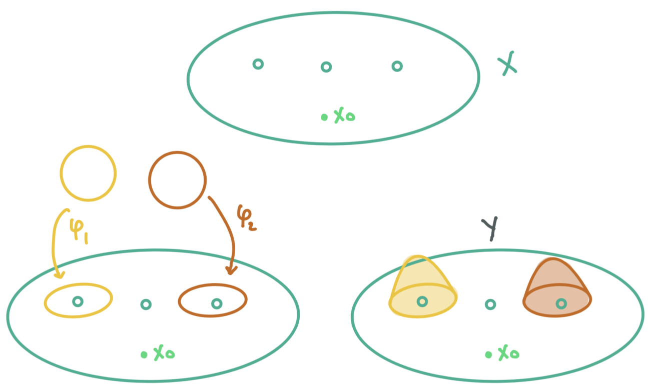

Figure 7: An instance of the spaces X and Y

   (b) **Definition (Cellular map).** A continuous map $f : X \to Y$ between CW complexes X and Y is called a *cellular map* if $f(X^n) \subseteq Y^n$ for all n.

   We will discuss the following important theorem later in the course.

   Theorem (Cellular approximation theorem). Every map $f : X \to Y$ of CW complexes is homotopic to a cellular map. If f is already cellular on a subcomplex $A \subseteq X$, the homotopy may be taken to be stationary on A.

   Use the cellular approximation theorem to deduce the following theorem.

   Theorem (The fundamental group of a CW complex is determined by its 2-skeleton). Let X be a path-connected CW complex. Let $\iota_1 : X^1 \to X$ and $\iota_2 : X^2 \to X$ be the inclusion of its 1-skeleton and 2-skeleton, respectively. The induced map $(\iota_1)_* : \pi_1(X^1) \to \pi_1(X)$ is surjective, and the induced map $(\iota_2)_* : \pi_1(X^2) \to \pi_1(X)$ is an isomorphism.

   (c) In a few sentences, summarize our conclusions from class on how to compute a presentation for π_1 of a CW complex.

2. **(Surfaces of different genera are not homeomorphic, or even homotopy equivalent).**

Definition (Connected sum). Let M_1 and M_2 be n-manifolds. The *connected sum* $M_1 \# M_2$ is the n-manifold constructed as follows. Delete an open n-ball B_i from M_i. Let $h : \partial B_1 \to \partial B_2$ be a homeomorphism. Then glue $M_1 \setminus B_1$ to $M_2 \setminus B_2$ via h:

$$M_1 \# M_2 = (M_1 \setminus B_1) \cup (M_2 \setminus B_2) \quad / \quad x \sim h(x) \quad \text{for all } x \in \partial B_1$$

Fact: If M_1 and M_2 are path-connected, then up to homeomorphism $M_1 \# M_2$ is independent of the choice of balls and homeomorphism h.

Definition (Closed genus-g surface). The *(closed) genus-*1 *surface* Σ_1 is a torus. In general the *(closed) genus-g surface* Σ_g is the connected sum of g tori.

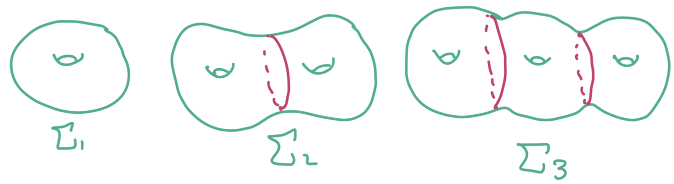

Figure 8: Surfaces of genus 1, 2, 3

(a) Briefly explain why the surface Σ_g can be realized as the quotient of a $2g$-gon by the edge identifications shown in Figure 9.

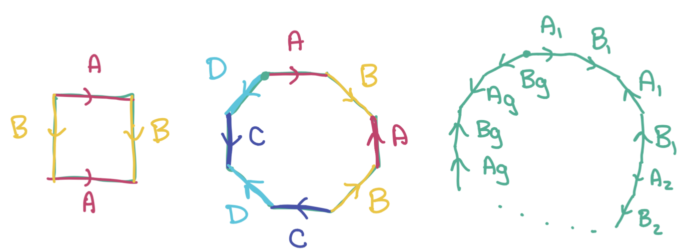

Figure 9: Σ_1, Σ_2, and Σ_g as quotients of polygons

Hint: See Figure 10.

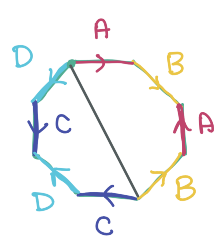

Figure 10

(b) Conclude that

$$\pi_1(\Sigma_g) = \langle a_1, b_1, a_2, b_2, \ldots a_g, b_g \mid [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] \rangle \qquad \text{where } [a, b] := aba^{-1}b^{-1}.$$

(c) Show that the abelianization of $\pi_1(\Sigma_g)$ is $\mathbb{Z}^{2g}$. Conclude that the surfaces Σ_g and Σ_h are not homotopy equivalent for any $g \neq h$.

3. **(Real and complex projective space: cell structure and fundamental group).**

   Definition (Real projective space). *Real n-dimensional projective space* $\mathbb{R}\mathrm{P}^n$ is the space of lines through the origin in $\mathbb{R}^{n+1}$. Specifically, $\mathbb{R}\mathrm{P}^n$ is the quotient space

$$\mathbb{R}\mathrm{P}^n \quad = \quad (\mathbb{R}^{n+1} \setminus \{0\}) \quad / \quad x \sim \lambda x \qquad \text{for any } \lambda \in \mathbb{R} \setminus \{0\}$$

   Definition (Complex projective space). *Complex n-dimensional projective space* $\mathbb{C}\mathrm{P}^n$ is the space of 1-dimensional subspaces [1] through the origin in $\mathbb{C}^{n+1}$,

$$\mathbb{C}\mathrm{P}^n \quad = \quad (\mathbb{C}^{n+1} \setminus \{0\}) \quad / \quad x \sim \lambda x \qquad \text{for any } \lambda \in \mathbb{C} \setminus \{0\}$$

   (a) Read Hatcher Example 0.4 and 0.6 for a description of CW complex structures on $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$. Summarize their construction here. (You may read the book as you write up your solution.) Illustrate $\mathbb{R}\mathrm{P}^1$, $\mathbb{R}\mathrm{P}^2$, and $\mathbb{C}\mathrm{P}^1$ with pictures.

   (b) Compute the fundamental groups of $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$ for all n. *Hint:* Assignment Problem 1.

Wellbeing

(This section is completely optional. This is a nudge to prioritize your wellbeing during the pandemic.)

1. **(Health comes first).** Make your sleep, fitness, nutrition, and other health goals a priority this week.

2. **(Gratitude).**

   (a) This week, identify four things that you are grateful for, big or small. Be specific. Take time to reflect on these things, and appreciate them. Think about how you would describe them, and their impact on you, to another person. There is an extensive psychology literature showing that "practicing gratitude" has persisting, measurable effects on people's self-assessed happiness levels.

   (b) Identify a reason you feel gratitude toward somebody in particular. Thank them.

[1] 1-dimensional complex subspsaces are sometimes called *complex lines*, even though they are 2 real dimensionnal