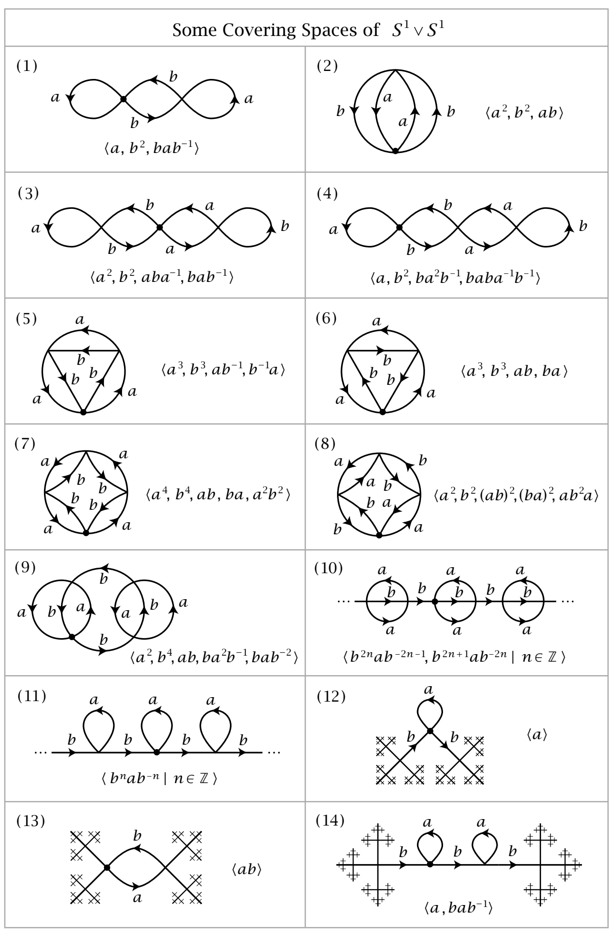
Terms and concepts covered: Covering spaces, lifting properties of covering spaces, classification of covering spaces.

Corresponding reading: Hatcher Ch 1.3

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let $p : \tilde{X} \to X$ be a covering map.
   (a) Let A be a subpace of X. Show that $p|_{p^{-1}(A)} : p^{-1}(A) \to A$ is a covering map.
   (b) Suppose B is a subspace of $\tilde{X}$. Is $p|_B : B \to p(B)$ necessarily a covering map?
2. Let $p : \tilde{X} \to X$ be a covering map. For $x_0 \in X$, show that $p^{-1}(x_0)$ is a discrete set.
3. Show that S^n is a cover of $\mathbb{R}\mathrm{P}^n$.
4. Consider the covers $p : \tilde{X} \to S^1 \vee S^1$ shown on Hatcher p58 (copied below).
   (a) For each cover, verify that $p_*(\pi_1(\tilde{X}))$ is the subgroup shown.
   (b) Consider the automorphism group of directed, labelled graphs $\tilde{X}$. This means the graph automorphisms that preserve the labels a and b and their orientations. For each cover $\tilde{X}$, determine whether this automorphism group acts transitively on vertices of $\tilde{X}$.



5. (a) For each n, construct an n-sheeted cover of S^1.
   (b) Show that there is no 3-sheeted cover of $\mathbb{RP}^2$.
   Hint: Assignment Problem 2. What are the subgorups of $\pi_1(\mathbb{RP}^2)$?

6. See Assignment Problem 3 for the definitions of simply-connected, locally simply-connected, and semi-locally simply-connected.
   (a) Show that a simply-connected space is semi-locally simply-connected.
   (b) Show that a locally simply-connected space is semi-locally simply-connected.
   (c) Verify that S^1 is locally simply-connected and semi-locally simply-connected but not simply-connected.
   (d) Verify that the Hawaiian earring is not semi-locally simply-connected.
   (e) Let C be the cone on the Hawaiian earring (in the sense of Homework #3 Problem 4(f)). Show that C is simply-connected and semi-locally simply-connected, but not locally simply-connected.

7. (a) Explain how we can identify the universal cover of S^1 constructed in 3 with our cover $\mathbb{R} \to S^1$.
   (b) Describe a cover of S^1 associated to each subgroup of $\mathbb{Z}$.

Assignment questions

(Hand these questions in!)

1. In this problem, we will prove that covering spaces satisfy the homotopy lifting property we encountered on Homework 2.

   Theorem (Covering maps have the homotopy lifting property). Let $p : E \to B$ be a covering map, and let $F_t : X \times I \to B$ be a homotopy of maps $X \to B$. Then given any lift $\tilde{F}_0 : X \to E$ of F_0, there exists a unique lift $\tilde{F}_t : X \times I \to E$ of F_t whose restriction to $t = 0$ is the lift $\tilde{F}_0$.

$$\begin{tikzcd} X \times \{0\} \cong X \arrow[r, "\tilde{F}_0"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "F_t"'] \arrow[ur, dashed, "\tilde{F}_t", "\exists !"'] & B \end{tikzcd}$$

   Let $p : E \to B$ be a covering map, and let $F_t : X \times I \to B$ be a homotopy of maps $X \to B$. Let $\tilde{F}_0 : X \to E$ be a lift of F_0. Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of B so that $p^{-1}(U_\alpha)$ can be decomposed as a disjoint union of open sets which are each mapped homeomorphically to U_α by p.
   Hint: For this proof, you may want to refer to the proof of Theorem 1.7 in Hatcher. Please put away the textbook as you write your solution.

   (a) We first address uniqueness. Suppose that $\gamma : I \to B$ is a path. Explain why there is a partition $0 = t_0 < t_1 < \ldots < t_n = 1$ of I so that for each i, $\gamma([t_i, t_{i+1}]) \subseteq U_i$ for some $U_i \in \mathcal{U}$.
   (b) Let $\tilde{b} \in p^{-1}(\gamma(0))$. Suppose that $\tilde{\gamma} : I \to E$ and $\tilde{\gamma}' : I \to E$ are two paths starting at $\tilde{b}$ lifting γ. Assume by induction that $\tilde{\gamma}|_{[0,t_i]} = \tilde{\gamma}'|_{[0,t_i]}$. Explain why necessarily $\tilde{\gamma}|_{[t_i,t_{i+1}]} = \tilde{\gamma}'|_{[t_i,t_{i+1}]}$. Conclude that there is a unique lift of γ starting at $\tilde{b}$.
   (c) Let $\tilde{F} : X \times I \to E$ be a homotopy lifting F_t and extending $\tilde{F}_0$. Explain why $\tilde{F}_t$ is unique.
   Hint: Consider $\tilde{F}|_{\{x\} \times I}$.
   (d) Now we address existence. Consider a point $x \in X$. Explain why there is a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ of I (depending on x) and a neighbourhood $N_x \subseteq X$ of x such that, for each i, $F(N_x \times [t_i, t_{i+1}]) \subseteq U_i$ for some U_i in $\mathcal{U}$. Notably N_x is independent of i.
   Hint: First fix $x_0 \in X$ and consider neighbourhoods $N_t \times (a_t, b_t)$ of (x_0, t) for each $t \in I$. Use compactness of I.

(e) Fix $x \in X$. Our next goal is to construct a lift $\tilde{F}^x$ of $F|_{N_x \times I}$ extending $\tilde{F}_0|_{N_x \times I}$. Assume by induction that $\tilde{F}^x$ is defined on $N_x \times [0, t_i]$. Describe how to extend $\tilde{F}^x$ to $N_x \times [0, t_{i+1}]$, and deduce that we can construct the desired lift $\tilde{F}^x$.
Hint: You may need to replace N_x by a smaller neighbourhood of x.

(f) For $x, y \in X$, explain why $\tilde{F}^x$ and $\tilde{F}^y$ must agree on $(N_x \cap N_y) \times I$. Explain how we can therefore combine the functions $\{\tilde{F}^x\}_{x \in X}$ to obtain a well-defined, continuous lift $\tilde{F}$ of the homotopy F extending $\tilde{F}_0$.

2. (a) Recall that a function N on a space X is *locally constant* if each $x \in X$ has a neighbourhood where N is constant. Show that a locally constant function is in fact constant on connected components of X.

(b) **Definition (Sheets of a cover).** Let X be a connected space. The number of *sheets* of a cover $p : \tilde{X} \to X$ is the cardinality of $p^{-1}(x)$ for a point $x \in X$.

Verify that the cardinality of $p^{-1}(x)$ is locally constant on X, and deduce that the number of sheets is well-defined for a cover of a connected space X.

(c) Let $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a cover with $X, \tilde{X}$ path-connected. Consider the function

$$\begin{aligned} \phi : \pi_1(X, x_0) &\longrightarrow p^{-1}(x_0) \\ [\gamma] &\longmapsto \tilde{\gamma}(1) \end{aligned}$$

where $\tilde{\gamma}$ is a lift of a representative γ starting at $\tilde{x_0}$. Show ϕ is well-defined.

(d) Show moreover that ϕ is well-defined on right cosets of $H = p_*(\pi_1(\tilde{X}, \tilde{x_0}))$, and so defines a function

$$\begin{aligned} \Phi : \pi_1(X, x_0) \bmod H &\longrightarrow p^{-1}(x_0) \\ H[\gamma] &\longmapsto \tilde{\gamma}(1) \end{aligned}$$

(e) Show that Φ is bijective. This proves the following theorem.

Theorem (Sheets of a cover and π_1). Let $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a cover with $X, \tilde{X}$ path-connected. The number of sheets of p is equal to the index of $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ in $\pi_1(X, x_0)$.

(f) The group $\mathbb{Z}^2$ has index-4 subgroups $(4\mathbb{Z} \times \mathbb{Z})$, $(\mathbb{Z} \times 4\mathbb{Z})$ and $(2\mathbb{Z} \times 2\mathbb{Z})$. Find a 4-sheeted covering map of the torus corresponding to each. No justification necessary.

3. **(Construction of the universal cover).** Throughout this question, when we refer to "path homotopy", or "homotopy classes of paths", we implicitly mean homotopy rel $\{0, 1\}$.
Hint: You may read Hatcher p 63-65 while you complete this question.

Definition (simply-connected). A space X is called 0-*connected* if it is path-connected. The space is *simply-connected* or 1-*connected* if it is path-connected and $\pi_1(X) = 0$.

Definition (Locally simply-connected). A space X is *locally simply-connected* if each point $x \in X$ has a neighbourhood basis of simply-connected open sets U.

Definition (Semi-locally simply-connected). A space X is called *semi-locally simply-connected* if every point $x \in X$ has a neighbourhood U such that the inclusion $U \hookrightarrow X$ induces the trivial map $\pi_1(U, x) \to \pi_1(X, x)$.

Observe that in the definition of semi-locally simply-connected, the neighbourhood U does not need to be simply-connected. A loop in U based at x may not contract to the constant map at x by a path homotopy in U, but it does contract to the constant map by a path homotopy in the larger space X. See Warm-Up Problem 6.
Fact: CW complexes are locally contractible, therefore locally simply-connected and semi-locally simply-connected.

(a) Suppose that a space X is not semi-locally simply-connected. Show that X cannot have a simply-connected cover.

(b) Suppose that X is simply-connected. Show that any two points in X are joined by a unique homotopy class of paths.

(c) Let $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a covering map. Use the lifting properties to show that there is a bijection between homotopy classes of paths in X starting in x_0 and homotopy classes of paths in $\tilde{X}$ starting at $\tilde{x_0}$.

(d) **Definition (The universal cover of X).** Let X be a path-connected, locally path-connected, semi-locally simply-connected space with basepoint x_0. Define

$$\tilde{X} = \{[\gamma] \mid \gamma \text{ a path in } X \text{ based at } x_0\}$$

where $[\gamma]$ is the homotopy class of the path γ. Define

$$\begin{aligned} p : \tilde{X} &\longrightarrow X \\ [\gamma] &\longmapsto \gamma(1) \end{aligned}$$

Let $\mathcal{U} = \{U \subseteq X \mid U \text{ is path-connected, open, and } \pi_1(U) \to \pi_1(X) \text{ is trivial}\}$. We topologize $\tilde{X}$ be defining a basis of open sets

$$U_{[\gamma]} = \{[\gamma \cdot \alpha] \mid \ U \in \mathcal{U}, \ \ \gamma \text{ a path from } x_0 \text{ to a point in } U, \ \ \alpha \text{ a path in } U \text{ with } \alpha(0) = \gamma(1) \ \}.$$

Show that the map p is well-defined and surjective.

(e) Show that $\mathcal{U}$ is a basis for the topology on X.

(f) Show that $U_{[\gamma]} = U_{[\gamma']}$ if $[\gamma'] \in U_{[\gamma]}$. Conclude that the sets $U_{[\gamma]}$ form the basis for a topology on $\tilde{X}$.

(g) Show that p is continuous. *Hint:* Show that, for $U \in \mathcal{U}$,

$$p^{-1}(U) = \bigcup_{\gamma \text{ a path from } x_0 \text{ to } U} U_{[\gamma]}.$$

(h) Show that $p|_{U_{[\gamma]}}$ is a homeomorphism to U. Deduce that p is a covering map.

(i) Show that $\tilde{X}$ is path-connected. *Hint:* For a point $[\gamma] \in \tilde{X}$, consider the path $t \mapsto [\gamma_t]$, where $[\gamma_t] \in \tilde{X}$ is the homotopy class of the path

$$\gamma_t(s) = \begin{cases} \gamma(s), & 0 \leq s \leq t \\ \text{constant function at } \gamma(t), & t \leq s \leq 1. \end{cases}$$

(j) Let $[x_0]$ denote the class of the constant path in X at x_0. Show that $\pi_1(\tilde{X}, [x_0]) = 0$.
Hint: We will prove in class that p_* is injective, so it suffices to show that its image is trivial. For a loop γ in X based at x_0, first show that $t \to [\gamma_t]$ is a lift of γ. Deduce that this lift is not a loop unless γ is trivial in $\pi_1(X, x_0)$.

(k) Give a brief/informal explanation of how we can identify the universal cover of $S^1 \vee S^1$ with the infinite tree shown in Figure 1.

4. **(The covering space of X associated to $H \subseteq \pi_1(X)$).** Let X be a path-connected, locally path-connected, semi-locally simply-connected space with basepoint x_0. Let $H \subseteq \pi_1(X, x_0)$ be a subgroup. Define X_H to be the quotient of the universal cover $\tilde{X}$ of X (Assignment Problem 3) by the equivalence relation

$$[\gamma] \sim [\gamma'] \quad \text{iff} \quad \gamma(1) = \gamma'(1) \text{ and } [\gamma \cdot \overline{\gamma'}] \in H.$$

Hint: You may read Hatcher Prop 1.36 while you complete this question.

(a) Verify that $\sim$ is well-defined and is an equivalence relation.

(b) Show that the covering map $p : \tilde{X} \to X$ factors through a map $p_H : X_H \to X$.

(c) Verify that p_H is a covering map.

(d) Show that the image of $(p_H)_*$ is H.

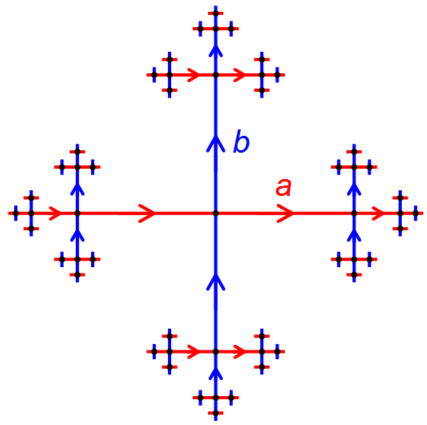


Figure 1: The universal cover of $S^1 \vee S^1$

Wellbeing

(This section is completely optional. This is a nudge to prioritize your wellbeing during the pandemic.)

1. **(Health comes first).** Re-evaluate and update your health goals. If there are goals you have not met, don't beat yourself up. Instead, consider: what are the obstacles, and how can you remove them going forward? Alternatively, can you replace the goal with a more achievable one?

2. **(Play).** This week, do something whimsical! Make a blanket fort, have a snowball fight, sing in the shower, build a snowman, have formal a tea party on your bedroom floor, put googly eyes on household items, have a dance party in the street.