Terms and concepts covered: lifting properties of covering spaces, classification of covering spaces

Corresponding reading: Hatcher Ch 1.3

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let $p : S^1 \to S^1$ be the 3-sheeted cover $e^{2\pi it} \mapsto e^{6\pi it}$. Explicitly describe, with pictures, the map Φ from Homework #5 Assignment Question 2(d) for this cover. Verify that the map is well-defined on the cosets of the subgroup $H = 3\mathbb{Z}$ in $\mathbb{Z}$.
2. A *torsion* group G is a group where every element has finite order. Show that the only group homomorphism from a torsion group G to a free abelian group $\mathbb{Z}^n$ is the zero map.
3. Let $p_1 : X_1 \to X$ and $p_2 : X_2 \to X$ be covering maps, and let $f : X_1 \to X_2$ be an isomorphism of covers (Assignment Problem 4 (a)).
   (a) Show that, for each $x \in X$, the map f defines a bijection between $p_1^{-1}(x)$ and $p_2^{-1}(x)$.
   (b) Show that f^{-1} is also an isomorphism of covers.
   (c) Verify that "isomorphism of covers" defines an equivalence relation on the covers of a fixed space X.
   (d) Fix a cover $p : \tilde{X} \to X$. Show that the isomorphisms $\tilde{X} \to \tilde{X}$ form a group under composition.
   (e) Compute the group of isomorphisms for the following covers.
      - $\mathbb{R} \to S^1$
      - The N-sheeted cover $S^1 \to S^1$
      - $S^n \to \mathbb{R}\mathrm{P}^n$
      - Your favourite covers from Hatcher's table on p58 (shown below).

Assignment questions

(Hand these questions in!)

1. **(Topology QR Exam, May 2016).** Let X be the complement of a point in the torus $S^1 \times S^1$.
   (a) Compute $\pi_1(X)$.
   (b) Show that every map $\mathbb{R}\mathrm{P}^n \to X$ is nullhomotopic for $n \geq 2$.
2. **(Topology QR Exam, Aug 2019).** Let $f : X \to Y, g : Z \to Y$ be connected coverings, where Y is a path-connected locally path-connected space. Let $X \times_Y Z = \{(x, z) \mid f(x) = g(z)\}$ with the subspace topology of the product topology. Let $p : X \times_Y Z \to Y$ be given by $(x, z) \mapsto f(x) = g(z)$.
   (a) Is p necessarily a covering?
   (b) Is $X \times_Y Z$ necessarily connected? (Prove your answers.)
3. (a) The following lemma is proved in Hatcher Lemma 1A.3.

   Lemma. Let X be a graph. Then every cover $\tilde{X}$ of X is a graph, with vertices and edges the lifts of vertices and edges, respectively, in X.

   Describe how to define a 1-dimensional CW complex structure on a cover $\tilde{X}$ of a graph X. You do not need to give point-set details. You may read Hatcher while you write your solution.

   (b) Prove the following theorem.

   Theorem. Every subgroup of a free group is free.

4. **(The Classification of Covering Spaces).**

   (a) **Definition (Isomorphism of covers).** Let $p_1 : X_1 \to X$ and $p_2 : X_2 \to X$ be covering maps. A continuous map $f : X_1 \to X_2$ is an *isomorphism of covers* if f is a homeomorphism and $p_1 = p_2 \circ f$.

   Use the lifting properties and uniqueness of lifts proved in class to prove the following proposition.

   Proposition (Uniqueness of the cover associated to a subgroup of $\pi_1(X)$). If X is path-connected and locally path-connected, then two path-connected covering spaces $p_1 : X_1 \to X$ and $p_2 : X_2 \to X$ are isomorphic via an isomorphism $f : X_1 \to X_2$ taking a basepoint $\tilde{x_1} \in p_1^{-1}(x_0)$ to a basepoint $\tilde{x_2} \in p_2^{-1}(x_0)$ if and only if

   $$(p_1)_*(\pi_1(X_1, \tilde{x_1})) = (p_2)_*(\pi_1(X_2, \tilde{x_2})).$$

   (b) Deduce the following important theorem, which is the culmination of your work on this and the previous assignment.

   Theorem (The classification of (based) covering spaces). Let X be path-connected, locally path-connected, and semi-locally simply-connected. Then there is a bijection between the set of basepoint-preserving isomorphism classes of path-connected covering spaces $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ and the set of subgroups of $\pi_1(X, x_0)$, obtained by associating the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ to the covering space $(\tilde{X}, \tilde{x_0})$.

   (c) Let $p : \tilde{X} \to X$ be a covering map with $\tilde{X}$ path-connected. Let $x_0 \in X$ and let $\tilde{x_0}, \tilde{x_1} \in p^{-1}(x_0)$. Analyze the change-of-basepoint map on $\tilde{X}$ to prove that $p_*(\pi_1(X, \tilde{x_0}))$ and $p_*(\pi_1(X, \tilde{x_1}))$ are conjugate subgroups of $\pi_1(X, x_0)$.

   (d) Prove the following variation on the classification theorem.

   Theorem (The classification of (unbased) covering spaces). Let X be path-connected, locally path-connected, and semi-locally simply-connected. Then there is a bijection between the set of isomorphism classes of path-connected covering spaces $p : \tilde{X} \to X$ and the set of conjugacy classes of subgroups of $\pi_1(X)$.

5. **(The action of $\pi_1(X, x_0)$ on the fibres).** Let X be a connected, locally path-connected, semi-locally simply-connected space. Let $p : \tilde{X} \to X$ be a covering map, and let α be a path in X. Define a map

   $$L_\alpha : p^{-1}(\alpha(0)) \to p^{-1}(\alpha(1))$$

   as follows: for a point $\tilde{x_0} \in p^{-1}(\alpha(0))$, lift α to the path $\tilde{\alpha}$ starting at $\tilde{x_0}$. Then $L_\alpha(\tilde{x_0}) = \tilde{\alpha}(1)$.

   (a) Explain why L_α only depends on the homotopy class of α rel $\{0, 1\}$.

   (b) Show that L_α is a bijection of sets. *Hint:* What is its inverse?

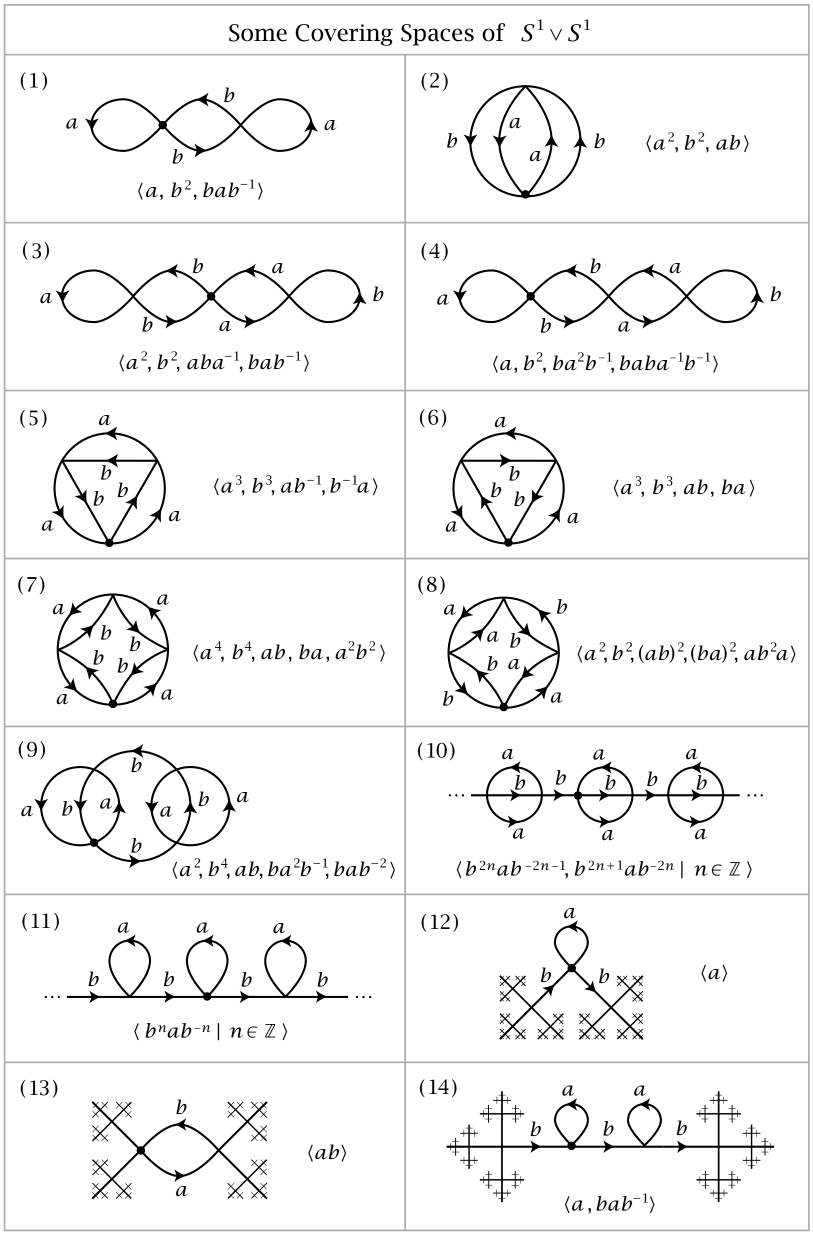
   (c) Show that $L_{\overline{\alpha \cdot \beta}} = L_{\overline{\alpha}} \circ L_{\overline{\beta}}$.
   (Note that we had to replace α by its inverse $\overline{\alpha}$ to make this relationship covariant).

   (d) Now let us restrict to classes $[\gamma] \in \pi_1(X, x_0)$. Conclude that the assignment

   $$\begin{aligned} \pi_1(X, x_0) &\longrightarrow \{\text{Permutations of } p^{-1}(x_0)\} \\ [\gamma] &\longmapsto L_{\overline{\gamma}} \end{aligned}$$

   defines a group action of $\pi_1(X, x_0)$ on the set $p^{-1}(x_0)$.

   (e) Choose five covers $\tilde{X}$ of $S^1 \vee S^1$ from Hatcher's table from p58 (copied below). Describe the permutation on the vertices of $\tilde{X}$ defined by the generator a, and the permutation defined by the generator b. No justification necessary; just state your answer.



(f) Recall the map Φ defined in Homework 5 Assignment Problem 2(d)

$$\begin{aligned} \Phi : \pi_1(X, x_0) \text{ mod } H &\longrightarrow p^{-1}(x_0) \\ H[\gamma] &\longmapsto \tilde{\gamma}(1) \end{aligned}$$

that defined a bijection between $p^{-1}(x_0)$ and the right cosets of $H = p_*(\pi_1(\tilde{X}, \tilde{x_0})$. Show that the group action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$ defined above corresponds to the usual action of $\pi_1(X, x_0)$ on the right cosets by right multiplication,

$$[\gamma] \cdot (H[\beta]) = H[\beta \cdot \overline{\gamma}]$$

(g) Deduce that, if H is normal in $\pi_1(X, x_0)$, the action of $\pi_1(X, x_0)$ induces a well-defined action by the quotient group $\pi_1(X, x_0)/H$.

In fact, we can reconstruct the cover $p : \tilde{X} \to X$ from the action of $\pi_1(X, x_0)$ on the fibre $F = p^{-1}(x_0)$ by taking a suitable quotient of $\tilde{X_0} \times F$, where $\tilde{X_0}$ is the universal cover. (This construction is described on p69-70 of Hatcher). Hatcher concludes that the n-sheeted covers of X are classifed by conjugacy classes of group homomorphisms from $\pi_1(X, x_0)$ to the symmetric grroup S_n.

6. (a) Let X be a wedge of n circles, so $\pi_1(X, x_0) = F_n$. Let $h : F_n \to G$ be a surjective group homomorphism. Explain how we could use the results of Assignment Problems 5 and 3 (a) to construct the graph $\tilde{X}$ covering X with fundamental group the subgroup ker$(h) \subseteq \pi_1(X, x_0)$. Explain moreover how we can use the cover $\tilde{X}$ to determine a free generating set for ker(h).

   (b) **(Topology QR Exam, May 2017).** Let F be the free group on a, b. Let $G = \{1, x, x^2\}$ be the cyclic group on three generators written multiplicatively. Let $h : F \to G$ be a homomorphism which sends $a \mapsto x, b \mapsto x^2$. Find free generators of Ker(h).

   (c) **(Topology QR Exam, Jan 2017).** Let F be the free group on a, b. Let G be a symmetric group (=group of all permutations) on three elements, and let $x, y \in G$ be elements of order 2 and 3, respectively. Let $h : F \to G$ be a homomorphism which sends $a \mapsto x, b \mapsto y$. Find free generators of Ker(h).

Wellbeing

(This section is completely optional. This is a nudge to prioritize your wellbeing during the pandemic.)

1. **(Health comes first).** Make your health and happiness goals a priority this week.

2. **(Sign off).** This week, re-evaluate how and how much you use social media. Try signing off for a day! Study results are mixed, but some research suggests that use of social media platforms (like Facebook, Twitter, and Instagram) increases feelings of loneliness and anxiety, and decreases life satisfaction. Could you use this time for reciprocal, face-to-face interactions instead?

3. **(Reaching out).** Do you know someone who is living alone during the pandemic? Give them a call.